Nonperturbative effect in noncritical string theory and T-duality

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nonperturbative formulation of string theory?

candidates: matrix model, SFT, $\cdots$:so far unsuccessful

noncritical string theory:

nonpert. formulation via matrix model, SFT $\sim$ loop eq. (SD eq.)

$$\Downarrow$$

1. **How does the nonperturbative effect look like in these formalisms?**
2. **Its nature under duality in these formalisms**

These study should be important for understanding the nonperturbative formulation of critical string theory.

David

Hanada, Hayakawa, Ishibashi, Kawai, T.K., Matsuo and Tada

Let us consider the one-matrix model

$$S = N\mathrm{tr}V(\phi), \qquad V(x) = \frac{1}{2}\phi^2 - \frac{g}{3}\phi^3,$$

where ϕ: $N \times N$ Hermitian matrix. The effective action for the eigenvalues λ_i $(i = 1, \cdots, N)$:

$$S_{\mathrm{eff}} = -\sum_{i<j} \log(\lambda_i - \lambda_j)^2 + N \sum_i V(\lambda_i).$$

Hereafter we consider a situation in which a single eigenvalue (λ_N) is separated from others. Then the partition function is expressed as

$$Z_N = \int dx \int d\lambda_1 \cdots d\lambda_{N-1} \left(\prod_{i=1}^{N-1} (x - \lambda_i)^2 \right) \times \Delta^{(N-1)}(\lambda_1, \cdots, \lambda_{N-1})^2 e^{-N\sum_{i=1}^{N-1} V(\lambda_i)} e^{-NV(x)}.$$

Therefore, Z_N is given as

$$Z_N = Z'_{N-1} \int dx \left\langle \det(x - \phi')^2 \right\rangle' e^{-NV(x)} \equiv Z'_{N-1} \int dx e^{-NV_{\mathrm{eff}}(x)},$$

where quantities with the prime are those in $(N-1)\times(N-1)$ matrix model except that $(N-1)$ is replaced by N in the exponent:

$$Z'_{N-1} = \int d\phi' e^{-N\mathrm{tr}V(\phi')}, \qquad \langle O\rangle' = \frac{1}{Z'_{N-1}}\int d\phi' O e^{-N\mathrm{tr}V(\phi')},$$
$$\phi' \ : \ (N-1)\times(N-1) \text{ matrix.}$$

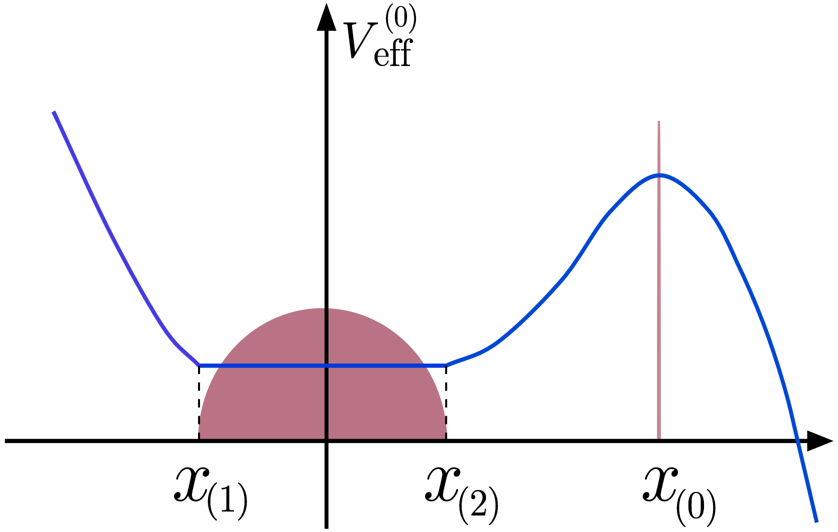
$V_{\mathrm{eff}}(x)$: effective potential for a single eigenvalue in question

In the large-N limit, V_{eff} becomes

$$\begin{aligned} V_{\mathrm{eff}}^{(0)}(x) &= -\frac{1}{N}\log\left\langle\det(x-\phi)^2\right\rangle + V(x) = -\left\langle\frac{1}{N}\mathrm{tr}\log(x-\phi)^2\right\rangle + V(x) \\ &= -2\mathrm{Re}\int_{x_*}^{x} dx' R(x') + V(x) = -\mathrm{Re}\int_{x_*}^{x} dx'\sqrt{V'(x')^2+p(x')}, \end{aligned}$$

where x_* fixes the origin of $V_{\mathrm{eff}}^{(0)}$, and $R(x)$: the resolvent given by

$$R(x) = \left\langle\frac{1}{N}\mathrm{tr}\frac{1}{x-\phi}\right\rangle = \frac{1}{2}\left(V'(x)+\sqrt{V'(x)^2+p(x)}\right).$$



Note: force=0 in $[x_{(1)}, x_{(2)}]$ (plateau) $\Leftarrow$ real part
$[x_{(1)}, x_{(2)}]$: support of the eigenvalue distribution
Matrix model instanton lies at $x = x_{(0)}$.
In the double scaling limit the height of the potential barrier becomes

$$N(V^{(0)}_{\rm eff}(x_{(0)}) - V^{(0)}_{\rm eff}(x_{(2)})) = \frac{8\sqrt{3}}{5} t^{\frac{5}{4}} = S_{\rm inst}.$$

Roughly, $S_{\rm inst} \sim \int 2 \times$ disk amp.

Contribution to the free energy

Hanada, Hayakawa, Ishibashi, Kawai, T.K., Matsuo and Tada

$$Z_N = \int d\lambda_1 \cdots d\lambda_N G, \quad G = \Delta^{(N)}(\lambda_1, \cdots, \lambda_N)^2 \exp\left(-N \sum_{i=1}^{N} V(\lambda_i)\right).$$

Then $Z_N = Z_N^{(0-\text{inst.})} + Z_N^{(1-\text{inst.})} + Z_N^{(2-\text{inst.})} + \cdots$,

k-instanton sector: k eigenvalues are separated from others.

Let us consider the 1-instanton sector:

$$\begin{aligned} Z_N^{(1-\text{inst.})} &= N \int_{x<x_{(1)}, x_{(2)}<x} dx \int_{x_{(1)} \le \lambda_i \le x_{(2)} \ (i \ne N)} \prod_{i=1}^{N-1} d\lambda_i G(x, \lambda_1, \cdots, \lambda_{N-1}) \\ &= N Z'^{(0-\text{inst.})}_{N-1} \int_{x<x_{(1)}, x_{(2)}<x} dx e^{-V_{\text{eff}}(x)}, \end{aligned}$$

Note: N: number of ways of specifying the isolated eigenvalue.

We can drop Z'_{N-1} by taking the ratio to $Z_N^{(0-\text{inst.})}$ as

$$\mu \equiv \frac{Z_N^{(1-\text{inst.})}}{Z_N^{(0-\text{inst.})}} = N \frac{\int_{x<x_{(1)}, x_{(2)}<x} dx e^{-N V_{\text{eff}}(x)}}{\int_{x_{(1)}<x<x_{(2)}} dx e^{-N V_{\text{eff}}(x)}},$$

which is the chemical potential of the instanton:

$$F = F^{(0-\text{inst.})} + \mu.$$

Instanton as D-instanton

Hanada, Hayakawa, Ishibashi, Kawai, T.K., Matsuo and Tada

Matrix model instanton = D-instanton in $c = 0$ string theory

- **it gives open boundaries to the world sheet:**

$$Z_N = \int dx \int d\lambda_1 \cdots d\lambda_{N-1} \left(\prod_{i=1}^{N-1} (x - \lambda_i)^2 \right) \times \Delta^{(N-1)}(\lambda_1, \cdots, \lambda_{N-1})^2 e^{-N\sum_{i=1}^{N-1} V(\lambda_i)} e^{-NV(x)}$$

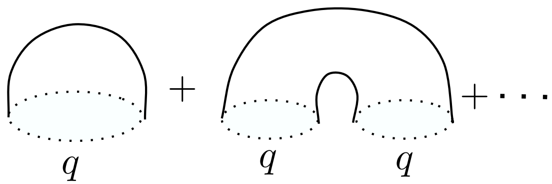
$$\sim \int dx \int d\phi' dq d\bar{q}\; e^{-S^{(1-\mathrm{inst.})} - NV(x)},$$

$$S^{(1-\mathrm{inst.})} = N\mathrm{tr}V(\phi') + \sum_{i=1,2} \bar{q}_i(\phi' - x)q_i,$$

ϕ' : $(N-1) \times (N-1)$ matrix,

q_i : Grassmann odd, fundamental rep..

$$\Downarrow$$

$\Delta Z_N^{(1-\mathrm{inst.})} =$ 

which accounts for the factor 2.

- **disk amp. in the fixed instanton background:**

Generically, the resolvent is given by

$$R(z) = \frac{1}{2}\left(V'(z) + \sqrt{V'(z)^2 + p(z)}\right), \quad p(z) = az + b.$$

irrespective of the instanton sector.
Thus b distinguishes the instanton sector
0-instanton sector: $p(z) = p_0(z)$
1-instanton sector: $p(z) = p_0(z) + c$

$$\begin{aligned} R(z) &= \frac{1}{2}\left(\sqrt{V'(z)^2 + p_0(z) + c} + V'(z)\right) \\ &= R_0(z) + \frac{c}{4}\frac{1}{\sqrt{V'(z)^2 + p_0(z)}} + \cdots, \end{aligned}$$

where $R_0(z)$: resolvent in the 0-inst. sector. Therefore, the contribution of the instanton to the resolvent is given by

$$\Delta R^{(1-\text{inst.})}(z) = \frac{c}{4}\frac{1}{\sqrt{V'(z)^2 + p_0(z)}} + \cdots .$$

value of c?

Note that the eigenvalue density $\rho(x)$ is given as

$$\rho(x) = -\frac{1}{\pi}\mathrm{Im}R(x+i0).$$

$\Delta\rho(z)$ corresponding to $\Delta R(z)$ should reflect the instanton. Thus we require in the 1-inst. sector

$$\int_{x_{(0)}-\epsilon}^{x_{(0)}-\epsilon} dx\, \Delta\rho(x) = \frac{1}{N},$$

which yields

$$c = \frac{8}{N} W_0'(x_{(0)}),$$

where

$$W_0(z) = \frac{1}{2}\sqrt{V'(z)^2 + p_0(z)}. \Rightarrow \Delta R^{(1-\mathrm{inst.})}(z) = \frac{1}{N}\frac{W_0'(x_{(0)})}{W_0(z)} + \cdots .$$

In the double scaling limit, the total disk amp. becomes

$$\tilde{w}(\zeta) + \frac{3\sqrt{3}}{8}\frac{t^{\frac{1}{4}}}{\tilde{w}(\zeta)}, \quad \text{where} \quad \tilde{w}(\zeta) = \left(\zeta - \tfrac{1}{2}\sqrt{t}\right)\sqrt{\zeta + \sqrt{t}}.$$

: 1-instanton effect

This completely agrees with the Liouville theory result using ZZ-boundary state.

Note: c: $\mathcal{O}(1/N)$, but gives a finite effect in the double scaling limit.

What we did so far:

- identify D-instanton in the matrix model
- $S_{\rm inst}$
- loop amp. in the fixed instanton background

How about the chemical potential?? Namely, weight of instanton itself

Chemical potential of instanton

Hanada, Hayakawa, Ishibashi, Kawai, T.K., Matsuo and Tada

$$\mu = \frac{Z_N^{(1-\text{inst.})}}{Z_N^{(0-\text{inst.})}} = N \frac{\int_{x<x_{(1)},x_{(2)}<x} dx \left\langle \det(x-\phi)^2 \right\rangle^{(0-\text{inst.})} e^{-NV(x)}}{\int_{x_{(1)}<x<x_{(2)}} dx \left\langle \det(x-\phi)^2 \right\rangle^{(0-\text{inst.})} e^{-NV(x)}},$$

1st question: finite? inst. survives in $N \to \infty$?

numerator

For $x < x_{(1)}, x_{(2)} < x$, $\det(x-\phi) = \exp\left(\mathrm{tr}\log(x-\phi)\right)$, thus

$$\left\langle \det(x-\phi)^2 \right\rangle = \exp\left[2\left\langle \mathrm{tr}\log(x-\phi)\right\rangle_c + \frac{1}{2}\left\langle \left(\mathrm{tr}\log(x-\phi)\right)^2 \right\rangle_c + \cdots\right].$$

Using

$$\begin{aligned} \left\langle \mathrm{tr}\log(x-\phi)\right\rangle_c &= \left\langle \mathrm{tr}\log(x-\phi)\right\rangle_{\text{disk}} + \mathcal{O}\left(\frac{1}{N}\right), \\ \left\langle \left(\mathrm{tr}\log(x-\phi)\right)^2 \right\rangle_c &= \left\langle \left(\mathrm{tr}\log(x-\phi)\right)^2 \right\rangle_{\text{cylinder}} + \mathcal{O}\left(\frac{1}{N^2}\right), \end{aligned}$$

$$
\begin{aligned}
&\left\langle \det(x-\phi)^2 \right\rangle e^{-NV(x)} \\
&= \exp\left[-N\left(-2\left\langle \frac{1}{N}\mathrm{tr}\log(x-\phi)\right\rangle_{\text{disk}} + V(x)\right)\right. \\
&\qquad \left. -\left(-2\ \left\langle (\mathrm{tr}\log(x-\phi))^2\right\rangle_{\text{cylinder}}\right) + \mathcal{O}\left(\frac{1}{N}\right)\right] \\
&= \exp\left(-N\, V_{\text{eff}}^{(0)}(x) - V_{\text{eff}}^{(1)}(x) + \mathcal{O}\left(\frac{1}{N}\right)\right) .
\end{aligned}
$$

$V_{\text{eff}}^{(0)}$ agrees with the previous one.

Then the integration in the numerator can be evaluated via the saddle point method which is valid in the large-N limit as

$$
\begin{aligned}
N\int_{x<x_{(1)},x_{(2)}<x} dx e^{-NV_{\text{eff}}(x)} &= \underbrace{e^{-NV_{\text{eff}}^{(0)}(x_{(0)})}}_{e^{\mathcal{O}(N)}} \\
&\times \underbrace{Ne^{-V_{\text{eff}}^{(1)}(x_{(0)})}\int dx e^{-\frac{N}{2}V_{\text{eff}}^{(0)\prime\prime}(x_{(0)})(x-x_{(0)})^2}}_{e^{\mathcal{O}(\log N)+\mathcal{O}(1)}}
\end{aligned}
$$

Known result of the cylinder amplitude yields

$$
\begin{aligned}
V^{(1)}_{\rm eff}(x) &= -2\left\langle\left(\mathrm{tr}\ln(x-\phi)\right)^2\right\rangle_{\rm cylinder} \\
&= -2\ln\left(1+\frac{x-\frac{x_{(1)}+x_{(2)}}{2}}{\sqrt{(x-x_{(1)})(x-x_{(2)})}}\right)+2\ln 2\,.
\end{aligned}
$$

: dependent only on $x_{(1)}$ and $x_{(2)}$.

denominator

leading: $\int_{x_{(1)}}^{x_{(2)}} e^{-NV_{\rm eff}^{(0)}(x)}dx \propto e^{-NV_{\rm eff}^{(0)}(x_{(2)})}$

$\longrightarrow$ fixes the origin of $V_{\rm eff}^{(0)}$ at $x = x_{(2)}$

$\longrightarrow$ $S_{\rm inst} = N(V_{\rm eff}^{(0)}(x_{(0)}) - V_{\rm eff}^{(0)}(x_{(2)}))$, as expected

However, cylinder amplitude $V_{\rm eff}^{(1)}$ is ill-defined, diverges at $x = x_{(2)}$.

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N-dependence of the cylinder amp. is necessary. The point is that whether it cancels the overall N (i.e. $V_{\rm eff}^{(1)} = -\log N + \text{finite}$).

$$\Downarrow$$

compute $\langle \det(x-\phi)^2 \rangle$ for $^\forall x$ up to $\mathcal{O}(1/N)$ in the exponent:

$$\langle \det(x-\phi)^2 \rangle = \exp\left[NC_0(x) + \log NC_1(x) + C_2(x) + \mathcal{O}\left(\frac{1}{N}\right)\right]$$

key: $D_N(x) = \langle \det(x-\phi)^2 \rangle_N$ satisfies a recursion formula

$$D_N(x) = P_N(x)^2 + r_N D_{N-1}(x)^2,$$

$P_N(x)$: orthogonal polynom, $xP_n(x) = P_{n+1}(x) + s_n P_n(x) + r_n P_{n-1}$.

$D_N(x)$ can be expressed in terms of $P_n(x)$, r_n as

$$D_N(x) = P_N(x)^2 + r_N\, P_{N-1}(x)^2 + \cdots + r_N \cdots r_1\, P_0(x)^2 \,.$$

$$\Downarrow$$

compute $P_n(x), r_n$ $(n \le N)$ up to $\mathcal{O}(1/N)$

By analyzing the recursion eq. for $P_n(x)$ and WKB-like consideration, we have obtained a nice expression for the asymptotic form of $P_n(x)$ with $n/N \sim \mathcal{O}(1)$ (even inside the cut!) as $\exp(\mathcal{O}(N) + \mathcal{O}(\log N) + \mathcal{O}(1))$, which enables us to obtain the explicit form of $V(x)$ up to $\mathcal{O}(1/N)$.

final result:

Outside the cut;

$$V_{\text{eff}}(x) = V_{\text{eff}}^{(0)}(x) - \frac{1}{N}\left\{-2\log\left[\frac{k^{(0)}(x,1)}{q(x,1)}\right]\right\},$$

Inside the cut;

$$V_{\text{eff}}(x) = V_{\text{eff}}^{(0)}(x) - \frac{1}{N}\log\left[\frac{N}{2}(x_{(2)} - x_{(1)})\pi\rho(x)\right].$$

Another derivation of the denominator

Ishibashi and Yamaguchi

The asymptotic form of $P_n(x)$ and the explicit form of $V_{\rm eff}(x,y)$ up to $\mathcal{O}(1/N)$ must be useful even for the future study.
However, as long as the evaluation of the denominator is concerned, the detailed analysis of the orthogonal polynomials is not necessary.

Let us consider a generic potential case

$$Z_N = \int d\phi e^{-\frac{N}{g^2}{\rm tr}V(\phi)}.$$

The denominator is given by

$$\int_{x_{(1)}}^{x_{(2)}} dx e^{-\frac{N}{g^2}V_{\rm eff}(x)} = \frac{Z_N^{(0-{\rm inst.})}}{Z'^{(0-{\rm inst.})}_{N-1}}.$$

However, $e^{-\frac{N}{g^2}V_{\rm eff}(x)} \propto \rho(x)$ is quite small outside $[x_{(1)}, x_{(2)}] \to$ we can replace the interval with $(-\infty, \infty)$ within the error $\mathcal{O}(1/N)$. Thus

$$\int_{x_{(1)}}^{x_{(2)}} dx e^{-\frac{N}{g^2}V_{\rm eff}(x)} = \frac{C_N \int d\phi e^{-\frac{N}{g^2}{\rm tr}V(\phi)}}{C_{N-1}\int d\phi' e^{-\frac{N-1}{g'^2}{\rm tr}V(\phi')}},$$

where $\frac{N-1}{g'^2} = \frac{N}{g^2}$. We expand

$$\int d\phi e^{-\frac{N}{g^2}\mathrm{tr}V(\phi)} = \exp(N^2 F_0(g^2) + F_1(g^2) + \cdots),$$

and the measure factor C_N dependent on N and g^2 is fixed in such a way that $F_0(g^2) = 0$ for $V(\phi) = \phi^2/2$. Similarly,

$$\int d\phi' e^{-\frac{N}{g'^2}\mathrm{tr}V(\phi')} = \exp((N-1)^2 F_0(g'^2) + F_1(g'^2) + \cdots).$$

Using these,

$$\int_{x_{(1)}}^{x_{(2)}} dx e^{-\frac{N}{g^2}V_{\mathrm{eff}}(x)}$$
$$= \frac{C_N}{C_{N-1}} \exp\Big[(N-1)\left(2F_0(g'^2) + F_1(g'^2)\right)$$
$$+ \left(F_0(g'^2) + 2g'^2\partial_{g'^2}F_0(g'^2) + \frac{1}{2}g'^4\partial^2_{g'^2}F_0(g'^2)\right) + \mathcal{O}(\frac{1}{N})\Big],$$

which means that in order to evaluate the denominator, it is sufficient to find the large-N free energy $F_0(g^2)$ and C_N . The result is

$$\int_{x_{(1)}}^{x_{(2)}} dx e^{-\frac{N}{g^2}V_{\mathrm{eff}}(x)} = \frac{(N-1)}{2}(x_{(2)} - x_{(1)})\pi e^{2(N-1)R},$$

which completely agrees with the one obtained by the orthogonal polynomial, where

$$R = \frac{1}{N-1}\langle \mathrm{tr}\log(\beta - \phi')\rangle - \frac{1}{2g'^2}V(\beta),$$

which is nonuniversal. However, in the numerator we have

$$e^{-V_{\mathrm{eff}}(x)} \propto \exp\left[2(N-1)R + 2(N-1)\int_{\beta}^{x} dx' \left(-\frac{M(x')}{2g'^2}\sqrt{(x'-\alpha)(x'-\beta)}\right)\right],$$

Thus we find that the nonuniversal quantity R completely cancels out and only the universal part of the resolvent contributes to S_{inst}, which ensures $S_{\mathrm{inst}} \propto 1/g_s$.

As we will see, this method can be also applied to the two-matrix model case where the orthogonal polynomials become quite complicated.

Anyway, in the double scaling limit we thus obtain

$$\mu = \frac{Z_N^{(1-\text{inst.})}}{Z_N^{(0-\text{inst.})}} = \frac{i}{8 \cdot 3^{\frac{3}{4}} \sqrt{\pi} t^{\frac{5}{8}}} e^{-\frac{8\sqrt{3}}{5} t^{\frac{5}{4}}},$$

which is proved to be universal!!, i.e. indep. of details of $V(x)$.
Therefore the matrix model fixes universally even the coefficient of the chemical potential of the instanton.
On the other hand, detailed analysis of the Painlevé eq. shows that it determines $S_{\text{inst}} = \frac{8\sqrt{3}}{5} t^{\frac{5}{4}}$ uniquely, but the coefficient appears as an integration constant and cannot be determined by the Painlevé eq itself.
→ Painlevé eq. misses the universal quantity. Matrix model should fix a boundary condition for Painlevé eq.

It is shown that the nonperturbative effect is also universal in type 0 string theory defined by the one-matrix model with the double-well type potential. Kawai, T.K., Matsuo

Ishibashi, T.K., Yamaguchi

Let us discuss how the nonperturbative effect is described in the generic $c < 1$ string theory. For this purpose, consider the two-matrix model

$$Z_N = C_N \int dA dB \exp\left[-\frac{N}{g^2}\mathrm{tr}(U(A) + \tilde{U}(B) - AB)\right],$$

with U and $\tilde{U}$ being polynomials of degree p and q.

In terms of eigenvalues, Itzykson-Zuber, Mehta

$$Z_N = \int \Pi_{i=1}^N d\lambda_i d\mu_i \Delta^{(N)}(\lambda)\Delta^{(N)}(\mu) e^{-\frac{N}{g^2}\sum_{i=1}^N (U(\lambda_i)+\tilde{U}(\mu_i)-\lambda_i\mu_i)}.$$

Specifying $(x, y) = (\lambda_N, \mu_N)$,

$$\begin{aligned} Z_N &= \int dx dy \int \Pi_{i=1}^{N-1} d\lambda_i d\mu_i \Pi_{i=1}^{N-1}\left((x-\lambda_i)(y-\mu_i)\right)\Delta^{(N-1)}(\lambda)\Delta^{(N-1)}(\mu) \\ &\times\ e^{-\frac{N}{g^2}\left(\sum_{i=1}^{N-1}(U(\lambda_i)+\tilde{U}(\mu_i)-\lambda_i\mu_i)+U(x)+\tilde{U}(y)-xy\right)} \\ &= Z'_{N-1}\int dx dy\ \langle \det(x-A')\det(y-B')\rangle' e^{-\frac{N}{g^2}(U(x)+\tilde{U}(y)-xy)} \\ &\equiv Z'_{N-1}\int dx dy\ e^{-NV_{\mathrm{eff}}(x,y)}. \end{aligned}$$

As before, $Z_N^{(1-\mathrm{inst.})}$ is defined by

$$Z_N^{(1-\mathrm{inst.})} = N {Z'}_{N-1}^{(0-\mathrm{inst.})} \int_{(x,y)\notin\mathcal{S}} dxdy e^{-V_{\mathrm{eff}}(x,y)},$$

where $\mathcal{S}$ is the support of the eigenvalue distribution, and the chemical potential is obtained by

$$\mu \equiv \frac{Z_N^{(1-\mathrm{inst.})}}{Z_N^{(0-\mathrm{inst.})}} = \frac{N \int_{(x,y)\notin\mathcal{S}} dxdy e^{-V_{\mathrm{eff}}(x,y)}}{\int_{(x,y)\in\mathcal{S}} dxdy e^{-V_{\mathrm{eff}}(x,y)}}.$$

In the large-N limit, $V_{\mathrm{eff}}(x,y)$ becomes

$$V_{\mathrm{eff}}^{(0)}(x,y) = \frac{N}{g^2}\Big(U(x) + \tilde{U}(y) - xy\Big) - \langle \mathrm{tr}\log(x-A)\rangle - \langle \mathrm{tr}\log(y-B)\rangle\ .$$

From this, we get the saddle point eqs.:

$$y = U'(x) - g^2 \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{x-A}\right\rangle, \quad x = \tilde{U}'(y) - g^2 \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{y-B}\right\rangle,$$

which is known to be solved as $x = X(s)$, $y = Y(s)$ with a uniformazation parameter $s \in C \cup \{\infty\}$. They are expanded as

$$X(s) = \gamma s + \sum_{k=0}^{p-1} \frac{\alpha_k}{s^k}, \quad Y(s) = \frac{\gamma}{s} + \sum_{k=0}^{q-1} \beta_k s^k.$$

Results

numerator:

$$\int_{(x,y)\notin\mathcal{S}} dxdy e^{-V_{\rm eff}(x,y)}$$

$$= (s-\tilde{s})^{-1}(-\partial X(s)\partial Y(\tilde{s}))^{-1/2}\frac{2\pi g'^2}{N-1}\left[\frac{\partial Y(s)}{\partial X(s)}\frac{\partial X(\tilde{s})}{\partial Y(\tilde{s})}-1\right]^{-1/2}$$

$$\times \exp\left[\frac{N-1}{g'^2}\left(2R-\int_{\tilde{s}}^{s} ds' Y(s')\partial X(s')\right)\right],$$

where $x = X(s) = X(\tilde{s})$, $y = Y(s) = Y(\tilde{s})$ is the saddle point and

$$2R = \lim_{s\to\infty,\tilde{s}\to 0}\int_{\tilde{s}}^{s} ds' Y(s')\partial X(s') + X(\tilde{s})Y(\tilde{s}) + C(s,\tilde{s}).$$

denominator:

$$\int_{(x,y)\in\mathcal{S}} dxdy e^{-V_{\rm eff}(x,y)} = (2\pi)^{3/2}\gamma\sqrt{(N-1)g'^2}\exp\left[\frac{2(N-1)}{g'^2}R\right].$$

We find that the nonuniversal R again cancels out.

$S_{\rm inst} = \frac{N-1}{g'^2}\int_{\tilde{s}}^{s} ds' Y(s')\partial X(s')$ is basically given by

$$S_{\rm inst} \sim \oint_B y dx, \quad y = U'(x) - g^2\left\langle \frac{1}{N}{\rm tr}\frac{1}{x-A}\right\rangle,$$

namely, an integration along a B-cycle. Kazakov and Kostov

Therefore, only the universal part of the resolvent again contributes to $S_{\rm inst}$, which ensures $S_{\rm inst} \propto 1/g_s$.

In the double scaling limit, the chemical potential becomes

$$\mu = -\frac{1}{8C}\left(\frac{g_s}{\pi p q \xi^{p+q}}\right)^{1/2}\left(\sin\frac{\pi m}{p}\sin\frac{\pi n}{q}\right)^{-1}\left(\sin\left(\frac{\pi n}{q}+\frac{\pi m}{p}\right)\sin\left(\frac{\pi n}{q}-\frac{\pi m}{p}\right)\right)^{1/2}$$
$$\times\left((-1)^{m+n+q+1}C^{p+q}\sin\frac{\pi m q}{p}\sin\frac{\pi n p}{q}\right)^{-1/2}$$
$$\times\exp\left[-\frac{8(-1)^{m+n+q}C^{p+q}pq}{g_s(p^2-q^2)}\xi^{p+q}\sin\frac{\pi m q}{p}\sin\frac{\pi n p}{q}\right].$$

Note:

1. $S_{\rm inst}$ agrees with the Kazakov and Kostov's and Liouville theory result.

2. We can further determine the subleading order: $\mu = e^{-NS_{\rm inst}+C_1\log N+C_2}$ and prove that μ is universal!!

3. The Douglas equation fixes $S_{\rm inst}$ uniquely, but does not determine the coefficient.

Thus we conclude that the chemical potential of the instanton or non-perturbative effect up to this order is universal in all $c < 1$ noncritical string theory even if its coefficient cannot be determined by the Douglas equation itself.

Our result should be compared with the one obtained recently by SFT approach. **Fukuma, Irie and Seki**

It would be interesting to examine what kind of loop is condensed at each saddle point.

T.K. and Sugino

T-duality: special case of the Kramers-Wannier duality

Essentially $G_{\mu\nu} \to G^{-1}_{\mu\nu}$ (Fourier transf.):

$$
\begin{aligned}
S = & \frac{1}{4\pi}\int d^2z G(X)\partial\theta\bar{\partial}\theta \\
\leftrightarrow \quad S' = & \frac{1}{4\pi}\int d^2z (G^{-1}(X)V\tilde{V} + \theta(\partial\tilde{V} - \bar{\partial}V)) \\
\leftrightarrow \quad S_D = & \frac{1}{4\pi}\int d^2z G^{-1}(X)\partial\theta_d\bar{\partial}\theta_d, \quad V = \partial\theta_d, \quad V = \bar{\partial}\theta_d,
\end{aligned}
$$

which is nothing but the Kramers-Wannier duality.

T-duality at the nonperturbative level: c=1/2 string theory

The original $c = 1/2$ string theory (Ising model on the random surface) is defined as the double scaling limit of the two-matrix model:

$$S = \mathrm{tr}\left(\frac{1}{2}A^2 - \frac{g}{3}A^3 + \frac{1}{2}B^2 - \frac{g}{3}B^3 - cAB\right).$$

A, B: up and down spin on the random surface.

Let us perform the Kramers-Wannier transf. on the random surface.

Matrix model propagator $\leftrightarrow$ Boltzmann weight for Ising model:

$$\langle AA\rangle = \langle BB\rangle = Le^{\beta} : \text{same}, \quad \langle AB\rangle = Le^{-\beta} : \text{opposite}, \quad c = e^{-2\beta}, \quad L = \frac{\sqrt{c}}{1-c^2}.$$

Z_2 Fourier transf.:

$$\begin{aligned} e^{\beta} &= K(e^{\tilde{\beta}} + e^{-\tilde{\beta}}), \\ e^{-\beta} &= K(e^{\tilde{\beta}} - e^{-\tilde{\beta}}). \end{aligned}$$

It is easy to see that the new matrix X, Y defined as

$$X = \frac{A+B}{\sqrt{2}}, \quad Y = \frac{A-B}{\sqrt{2}},$$

have the desired propagator:

$$\langle XX\rangle = \frac{1}{1-c} = \frac{1}{\sqrt{1-c^2}}e^{\tilde{\beta}}, \quad \langle YY\rangle = \frac{1}{1+c} = \frac{1}{\sqrt{1-c^2}}e^{-\tilde{\beta}}.$$

Thus we arrive at the dual two-matrix model

$$S_D = \mathrm{tr}\left(\frac{1-c}{2}X^2 + \frac{1+c}{2}Y^2 - \frac{\hat{g}}{3}(X^3 + 3XY^2)\right).$$

Note that the partition function is the same, but the correlators are different:

$$\left\langle \frac{1}{N}\mathrm{tr}\frac{1}{x-A}\right\rangle + \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{x-B}\right\rangle \not\propto \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{x-X}\right\rangle.$$

<u>Fact</u>:

1. $\langle XX \rangle$ and $\langle YY \rangle$ denotes "stick" and "flip" of the dual spin, respectively.

2. T-duality is violated at higher genus!! Y-loop represents the boundary of domain of the dual spin and a global Y-loop along a nontrivial homology cycle cannot be described by the spin variable as in the original model. This reflects the fact that in the dual model there exist global winding modes, which do not exist in the original model. Actually, in solving the constraint, such a DOF appears in general at higher genus.

3. In the sphere approximation, both theory are the same. The constraint can be solved by introducing the dual spin.

4. The disk amplitudes with one handle are different as an evidence of the violation of the T-duality.

How about the nonperturbative effect?

T-duality maps the nonperturbative effect (ZZ-brane) in the original model into the one in the dual model?

original model

General argment before implies that S_{inst} can be computed as

$$S_{\mathrm{inst}} = N V_{\mathrm{eff}}^{(0)}(x_{(0)}, y_{(0)}) = -N \left(\int^{x_{(0)}} dx R_A^{\mathrm{univ}}(x) + \int^{y_{(0)}} dy R_B^{\mathrm{univ}}(y) \right),$$

where $(x_{(0)}, y_{(0)})$ is a saddle pt., and $R_A^{\mathrm{univ}}(x)$, $R_B^{\mathrm{univ}}(y)$ are the universal part of the resolvent for A and B, respectively.

known result:

$$(1,1)\ \text{boudary condition}\ :\ \frac{S_{\mathrm{inst}}}{\sqrt{F_0}} = \frac{4\sqrt{7}}{6} \leftarrow h_{1,1} = 0$$

$$(1,2)\ \text{boudary condition}\ :\ \frac{S_{\mathrm{inst}}}{\sqrt{F_0}} = \frac{8\sqrt{3}}{7} \leftarrow h_{1,2} = 1/16$$

$$(1,3)\ \text{boudary condition}\ :\ \frac{S_{\mathrm{inst}}}{\sqrt{F_0}} = \frac{4\sqrt{7}}{6} \leftarrow h_{1,3} = 1/2$$

dual model

Applying the Itzykson-Zuber Integral, we obtain

$$
\begin{aligned}
Z &= \int dX dY e^{-NS_D} \\
&\propto \int \Pi_{i=1}^N d\lambda_i d\mu_i \Delta^{(N)}(\lambda)^2 \Delta^{(N)}(\mu)^2 \\
&\quad \times \exp\left[-N\sum_i \left(\frac{1-c}{2}\lambda_i^2 + \frac{1+c}{2}\mu_i^2 - \frac{\hat{g}}{3}\lambda_i^3\right)\right] \frac{\det_{ij} e^{N\hat{g}\lambda_i\mu_j^2}}{\Delta^{(N)}(\lambda)\Delta^{(N)}(\mu^2)} \\
&\propto \int d\lambda_i d\mu_i \frac{\Delta(\lambda)\Delta(\mu)}{\Pi_{i>j}(\mu_i+\mu_j)} \\
&\quad \times \exp\left[-N\sum_i \left(\frac{1-c}{2}\lambda_i^2 + \frac{1+c}{2}\mu_i^2 - \frac{\hat{g}}{3}(\lambda_i^3 + 3\sum_i \lambda_i\mu_i^2)\right)\right] \\
&= \int dxdy \left\langle \frac{\det(x-X')\det(y-Y')}{\det(y+Y')} \right\rangle' e^{-N\left(\frac{1-c}{2}x^2+\frac{1+c}{2}y^2-\frac{\hat{g}}{3}(x^3+3xy^2)\right)} \\
&\equiv \int dxdy e^{-NV_{\mathrm{eff}}(x)},
\end{aligned}
$$

In the large-N limit,

$$\begin{aligned} V_{\rm eff}^{(0)}(x,y) &= \frac{1-c}{2}x^2 + \frac{1+c}{2}y^2 - \frac{\hat{g}}{3}(x^3+3xy^2) \\ &- \frac{1}{N}\left<{\rm tr}\log(x-X)\right> - \frac{1}{N}\left<{\rm tr}\log(y-Y)\right> + \frac{1}{N}\left<{\rm tr}\log(y+Y)\right> \\ &= \frac{1-c}{2}x^2 + \frac{1+c}{2}y^2 - \frac{\hat{g}}{3}(x^3+3xy^2) - \frac{1}{N}\left<{\rm tr}\log(x-X)\right>, \end{aligned}$$

due to Z_2 symmetry $Y \to -Y$. Therefore, only the universal part of the resolvent for X contributes to $S_{\rm inst}$. If we look for the saddle pt. of $V_{\rm eff}^{(0)}(x,y)$ as a real function, we find one with

$$\frac{S_{\rm inst}}{\sqrt{F_0}} = \frac{4\sqrt{7}}{6},$$

which is nothing but the nonperturbative effect known in the $c = 1/2$ string theory. We expect that if we look for another saddle pt. on the other sheets, we find all nonperturbative effects with $S_{\rm inst}$ known in the $c = 1/2$ string theory, because $S_{\rm inst}$ receives contributions only from the disk and sphere diagrams for which both models are equivalent. However, it is possible that the coefficients of the chemical potential are different, because the cylinder amplitude contributes to it and the

global Y-loop can affect it. If this is true, the coefficient of the chemical potential reflects the effect of the violation of the T-duality. Anyway, it would be interesting to compute the chemical potential with the coefficient in both models and compare them.

$$\Downarrow$$

implications to critical string theory