Logarithmic CFT and 2D Minimal Gravity

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work in progress

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Intro

2D Minimal Gravity

or Minimal String Theory (MST)

— interesting string ‘laboratory’ as a 2D string [Seiberg et.al]

is [2D Gravity + (p, q) minimal CFT matter]

$\Downarrow$ $\Downarrow$

local continuum approach	topological discrete approach
field rep. of the surface	discretisation of the surface
Liouville Gravity	Matrix Model

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+ Perturbation [Zamolodchikov]

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? $\Rightarrow$ algebraic approach $SL(2, \mathbf{R})$ WZNW?

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$\Updownarrow$ [Nakayama]

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$\Uparrow$ [YI]

underlying theory? Logarithmic CFT?

Intro

Worldsheet Action

$= $ 2d gravity (Liouville FT) $+$ (p,q) minimal CFT

[Polyakov '81][DDK '88'89] [BPZ '84]

$$c_{tot} = c_L + c_{p,q} = 26.$$

Interesting for:
— String Theory, LFT itself, CFT on Random Surfaces, Integrability ...

Brief History from Logarithms

- 2d gravity – gravitationally dressed CFT as Logarithmic CFT (LCFT) [Bilal, Kogan '94]
- A puncture-type operator of LFT appears in LCFT [Kogan '97]
- A logarithmic Liouville 4-pt function in $(p,q)=(4,3)$ MST [Yamaguchi '02]
- Chiral logarithmic 4-pt functions in the Coulomb gas [Ishimoto '03]
- A op-valued relation of logarithmic degenerate field [Zamolodchikov '03]

Aim:

- 2d gravity (LFT) $\underset{Relation}{\longleftrightarrow}$ LCFT.
- Logarithms in Minimal String Theory.
- Its relation to Matrix Models.

To do:

4-pt Liouville correlation functions of tachyons
in 2d (p, q) minimal gravity
a full correlation function
in Gravitational Ising Model
to see a relation to LCFT.

1. 4pt Func of Liouville Sector

1.1 Action, 'Tachyon' primary
1.2 Condition for integer s
1.3 Integral Formulas
1.4 Logarithms in 4pt func. of $s = 1$

based on [hep-th/0406262]

2. 4pt Func of Full theory

based on [JPS talk, etc.]

3. Logarithmic CFT?

2D minimal gravity: Action

MST (conformal gauge) $\longrightarrow$ LFT + Minimal CFT

We First Focus On This $\uparrow$

The Action on S^2 [David88, DK89]:

$$S_{\mathrm{L}}[\hat{g}, \phi] = \frac{1}{8\pi} \int d^2z \sqrt{\hat{g}} \left(\hat{g}^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - Q \hat{R} \phi + 4\mu \, e^{\alpha\phi} \right), \quad (1)$$

$\hat{g}_{\alpha\beta}$: the reference metric; $\hat{R}$: 2d scalar curvature

μ: the renormalised cosmological constant

$Q = -\alpha - \frac{2}{\alpha}$ with $\alpha = -\sqrt{\frac{2q}{p}}$ $\Longrightarrow$ LFT = CFT with $c_{\mathrm{L}} = 1 + 3Q^2$

[Curtright, Thorn '82].

2D minimal gravity: Tachyon

'Tachyon' $\equiv$ Liouville primary $\otimes$ (p,q) primary [DDK]:

$$O_{r,t} = \int d^2z\sqrt{\hat{g}}\, O_{r,t}(z,\bar{z})\,,$$
$$O_{r,t}(z,\bar{z}) = e^{\beta_{r,t}\phi(z,\bar{z})}\Phi_{r,t}(z,\bar{z}) \qquad (2)$$

with the on-shell condition $\Delta_{O_{r,t}} = 1$

$O_{r,t}(z,\bar{z})$: a gravitationally dressed operator.

$\Phi_{r,t}(z,\bar{z})$ is a Kac primary field of the (p,q) matter.

$e^{\beta_{r,t}\phi(z,\bar{z})}$ is a gravitationally dressing operator of the Liouville sector.

$\beta_{r,t}$ is fixed by the on-shell condition [Seiberg '90, Polchinski '91][2]:

$$\beta_{r,t} = (1-r)\frac{1}{\alpha} + (1+t)\frac{\alpha}{2}\,, \qquad (3)$$

These are common and basic ingredients...

[2] The conformal dim of $e^{\beta\phi(z,\bar{z})}$ is $h_\beta = -\frac{1}{2}\beta(\beta+Q)$.

4-pt Liouville Correlation Functions of Tachyons

[Di Francesco, Kutasov '91]

By integrating out the Liouville zero mode ϕ_0 $(\phi = \phi_0 + \tilde{\phi})$:

$$\left\langle \prod_{i=1}^{4} e^{\beta_i \phi(z_i, \bar{z}_i)} \right\rangle = \left(\frac{\mu}{2\pi}\right)^s \frac{\Gamma(-s)}{-\alpha} \tilde{G}_{\mathrm{L}}^{(s)},$$

$$\tilde{G}_{\mathrm{L}}^{(s)} = \left\langle \prod_{i=1}^{4} e^{\beta_i \tilde{\phi}(z_i, \bar{z}_i)} \left(\int d^2u \, e^{\alpha \tilde{\phi}(u, \bar{u})} \right)^s \right\rangle,$$

with the free field action of $\tilde{\phi}$. $\beta_i = \beta_{r_i, t_i}$ and $h_i = h_{\beta_{r_i, t_i}}$ for short, and

$$s = -\frac{1}{\alpha} \left(Q + \sum_{i=1}^{4} \beta_i \right). \tag{4}$$

Regularisation for $s \in \mathbf{Z}_+$: $\left(\frac{\mu}{2\pi}\right)^s \Gamma(-s) \longrightarrow \left(\frac{\mu}{2\pi}\right)^s \frac{(-1)^{s+1}}{\Gamma(s+1)} \ln \mu$.

◇ Then, we just need evaluate $\tilde{G}_{\mathrm{L}}^{(s)}$ as in [Dotsenko, Fateev '84 '85].

Liouville correlation function for $s \in \mathbf{Z}_{\geq 0}$

$$\tilde{G}_{\mathrm{L}}^{(s)} = \prod_{1\leq i<j\leq 4} |z_i - z_j|^{-2(h_i+h_j)+\frac{2}{3}h} \, |\xi|^{2(h_1+h_2)-\frac{2}{3}h-2\beta_1\beta_2}$$

$$\times \, |1-\xi|^{2(h_2+h_3)-\frac{2}{3}h-2\beta_2\beta_3} \, I^{(s)}(-\alpha\beta_1, -\alpha\beta_3, -\alpha\beta_2; -\tfrac{1}{2}\alpha^2; \xi, \bar{\xi}) \, , \quad (5)$$

where $h = \sum_{i=1}^{4} h_i$, $\xi = \frac{z_{12}z_{34}}{z_{13}z_{24}} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}$ and:

$$I^{(s)}(a,b,c;\rho;\xi,\bar{\xi}) = \int \prod_{i=1}^{s} d^2u_i \prod_{i=1}^{s} \left[|u_i|^{2a}|1-u_i|^{2b}|u_i-\xi|^{2c}\right] \prod_{1\leq i<j\leq s} |u_i-u_j|^{4\rho}. \quad (6)$$

— What are possible combinations for a particular s?
(sec 1.2)

— How to calculate $I^{(s\neq 0)}(\xi, \bar{\xi})$
(sec 1.3).

1.2 Four fields for $s \in \mathbf{Z}_{\geq 0}$

◇ Generally, $s \in \mathbf{Q}$ for coprime p, q.
$\Rightarrow$ Analytic continuation from $s \in \mathbf{Z}_{\geq 0}$.
c.f. 3pt func. in [Dorn, Otto '92]

◇ Each $\beta_i \leftrightarrow (r_i, t_i)$ lives on the conformal grid $G_{p,q}$.

$$G_{p,q} = \left\{ (r,t) \in \mathbf{Z}^2 | 0 < r < q, 0 < t < p \right\}.$$

For this not to be an empty set, we assume $p \geq 2, q \geq 2$.

Finding possible combinations of four fields for given s, p, q $\rightarrow$ a combinatorics problem on 8-dimensional lattice $(G_{p,q})^4$.

$$2s = \frac{p}{q}\left(\sum_i r_i - 2\right) - \left(\sum_i t_i + 2\right). \tag{7}$$

Kac Table, Conformal Grid $G_{p,q}$

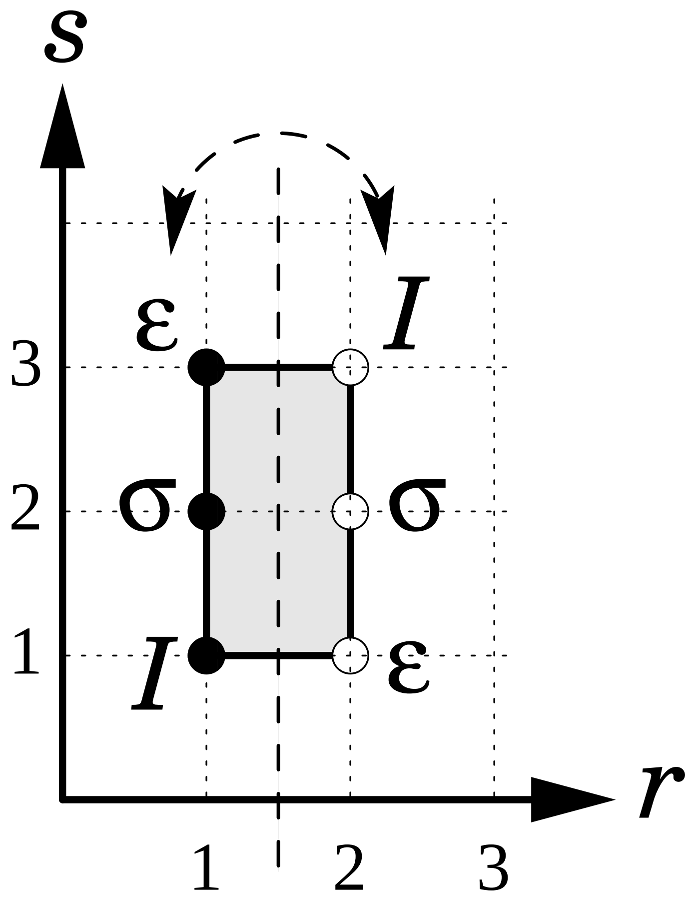

Figure 1: The Kac table for the Ising model ($c = 1/2$).

Four fields for $s \in \mathbf{Z}_{\geq 0}$ and $s = 1$

◇ For $s \in \mathbf{Z}_{\geq 0}$, eq. for r_i become dependent of t_i, and prescribes a 3-dim surface in 4-dim lattice.

$$\sum_i r_i = n \cdot q + 2 \qquad \text{for } n = 1, 2, 3.(for\ 4pt)$$
$$\sum_i t_i = n \cdot p - 2(1 + s). \tag{8}$$

$e.g.$ for $(p, q) = (4, 3)$ case, the following combinations give $s = 0$.

$$\langle \epsilon\epsilon\epsilon\tilde{I} \rangle \sim \langle \epsilon\epsilon\epsilon \rangle, \quad \langle \epsilon\epsilon\sigma\sigma \rangle.$$

◇ when $p \geq 4,\ q \geq 3,\ ^{\exists}$fields $s.t.$ $s = 1$:

$$\langle\ O_{r,t}\ \ O_{q-(r-1),p-(t+2)}\, O_{r,t}\ \ O_{q-(r-1),p-(t+2)}\ \rangle$$
$$\text{for } \ 2 \leq r \leq q-1,\ 1 \leq t \leq p-3. \tag{9}$$

· # of combinations is $(p-3)(q-2)$.

1.3 Integral expressions and explicit calculations for $s = 1$

By Dotsenko's formula [D88]:

$$\begin{aligned} I^{(1)}(a,b,c;0;\xi,\bar{\xi}) &= \int d^2u\,|u|^{2a}|1-u|^{2b}|u-\xi|^{2c}. \\ &= G_1\,|F_1(\xi)|^2 + G_2\,|F_2(\xi)|^2\,, \end{aligned}$$

where G_i's are coefficients and $F_i(\xi)$'s are two independent hypergeometric functions. **In LCFT cases, this form may be <u>indefinite</u>.**

◇ One way: the <u>differentiation</u> method [Ya '02, etc.]

◇ Another way: perform the same <u>analytic continuation</u> as in [D88], then express two of them in $|\xi - 1| < 1$:

$$I^{(1)}(a,b,c;0;\xi,\bar{\xi}) = -\sin(\pi a)\,I_2(\xi)\,I_3(\bar{\xi}) - \sin(\pi b)\,I_4(\xi)\,I_1(\bar{\xi})\,, \quad (10)$$

where

$$
\begin{aligned}
I_1 &\equiv \int_1^{\infty} du\, u^a (u-1)^b (u-\xi)^c \\
&= \frac{\Gamma(-1-a-b-c)\,\Gamma(1+b)}{\Gamma(-a-c)}\, {}_2F_1(-c, -1-a-b-c; -a-c; \xi)\,, \\
I_2 &\equiv \int_0^{\xi} du\, u^a (1-u)^b (\xi-u)^c \\
&= \frac{\Gamma(1+a)\,\Gamma(1+c)}{\Gamma(2+a+c)}\, \xi^{1+a+c}\, {}_2F_1(-b, 1+a; 2+a+c; \xi)\,, \\
I_3 &\equiv \int_{-\infty}^{0} du\, (-u)^a (1-u)^b (\xi-u)^c = \int_1^{\infty} du\, (u)^a (u-1)^b (u-(1-\xi))^c \\
&= \frac{\Gamma(-1-a-b-c)\,\Gamma(1+a)}{\Gamma(-b-c)}\, {}_2F_1(-c, -1-a-b-c; -b-c; 1-\xi)\,, \\
I_4 &\equiv \int_{\xi}^{1} du\, u^a (1-u)^b (u-\xi)^c = \int_0^{1-\xi} du\, (u)^b (1-u)^a (1-\xi-u)^c \\
&= \frac{\Gamma(1+b)\,\Gamma(1+c)}{\Gamma(2+b+c)}\, (1-\xi)^{1+b+c}\, {}_2F_1(-a, 1+b; 2+b+c; 1-\xi)\,. \qquad (11)
\end{aligned}
$$

Substituting the above into (10), we obtain:

$$
\begin{aligned}
I^{(1)}(a,b,c;0;\xi,\bar{\xi}) \;&=\; -\Gamma(-1-a-b-c)\,\Gamma(1+c) \\
\times \Big[&\sin(\pi a)\, U_{23}\, \xi^{1+a+c} \\
& \times {}_2F_1(-b,1+a;2+a+c;\xi)\, {}_2F_1(-c,-1-a-b-c;-b-c;1-\bar{\xi}) \\
& + \sin(\pi b)\, U_{41}\, (1-\xi)^{1+b+c} \\
& \times {}_2F_1(-a,1+b;2+b+c;1-\xi)\, {}_2F_1(-c,-1-a-b-c;-a-c;\bar{\xi}) \Big] \,,
\end{aligned}
\tag{12}
$$

with $U_{23} = \frac{[\Gamma(1+a)]^2}{\Gamma(2+a+c)\,\Gamma(-b-c)}$ and $U_{41} = \frac{[\Gamma(1+b)]^2}{\Gamma(2+b+c)\,\Gamma(-a-c)}$.

Note:

♣No-pole condtion: $a, b, c \notin \mathbf{Z}_-$.

♣ The analytic continuation is well-defined and exact when $(a+b+c) < -1$ and ξ is real. However, we assume that ξ can be analytically continued to $\mathbf{C}$ (or regard as a regularised expression).

Evaluation of $\tilde{G}_{\mathrm{L}}^{(s)}$ for $s = 1$

$$\tilde{G}_{\mathrm{L}}^{(1)} = |z_1 - z_3|^{-4h_1}|z_2 - z_4|^{-4h_2}|\xi(1-\xi)|^{-2\beta_1\beta_2} \times I^{(1)}(A, A, -1-A; 0; \xi, \bar{\xi}) \,. \tag{13}$$

where

$$I^{(1)}(A, A, -1-A; 0; \xi, \bar{\xi}) = \frac{(-1)^{1+r}\pi^2}{\sin\left(\pi(1+t)\frac{q}{p}\right)} \times \left\{ {}_2F_1(-A, 1+A; 1; \xi)\, {}_2F_1(-A, 1+A; 1; 1-\bar{\xi}) + (c.\,c.)\right\} . \tag{14}$$

and

$$\begin{aligned} -\alpha\beta_1 &\equiv A = -1 + r - (1+t)\frac{q}{p}\,, \\ -\alpha\beta_2 &= -r + (1+t)\frac{q}{p} = -1 - A\,, \end{aligned} \tag{15}$$

♣ Crossing symmetric and Monodromy invariant as usual.

♣ Most conditions for this calculation are satisfied by $A \notin \mathbf{Z}$.

1.4 Result with Logarithms

This correlation function is logarithmic

since

$$
\begin{aligned}
&{}_2F_1(-A, 1+A; 1; \xi) \\
&= \frac{\sin(\pi A)}{\pi} \Bigg[\ln(1-\xi)\, {}_2F_1(-A, 1+A; 1; 1-\xi) \\
&\quad - \sum_{n=0}^{\infty} \frac{(-A)_n (1+A)_n}{(n!)^2} (1-\xi)^n \left\{ 2\psi(n+1) - \psi(-A+n) - \psi(1+A+n) \right\} \Bigg],
\end{aligned} \tag{16}
$$

where $(a)_n$ is the Pochhammer symbol and $\psi(x) = \frac{\partial}{\partial x} \ln(\Gamma(x))$.
Hence,

2D minimal gravity

or Minimal string theories are logarithmic.

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3. Logarithmic CFT?

4-pt function of Full Theory

[JPS talk, work in progress]

Recall 'tachyon' primary field (2):

$$O_{r,t} = \int d^2z \sqrt{\hat{g}}\, e^{\beta_{r,t}\phi(z,\bar{z})} \Phi_{r,t}(z,\bar{z})\,, \tag{17}$$

We show the form of 4-pt func. of the full theory:

$$\begin{aligned}
&\langle\, O_{2,1}\, O_{q-1,p-3}\, O_{2,1}\, O_{q-1,p-3} \rangle \\
= \quad & \int \prod_{i=1}^{4} d^2 z_i \,\langle \prod_{i=1}^{4} O_i(z_i,\bar{z}_i) \rangle_{L+M+gh} \\
= \quad & \left(\frac{\mu}{2\pi}\right)^s \frac{\Gamma(-s)}{|\alpha|} \int \prod_{i=1}^{4} d^2 z_i\; \tilde{G}_L^{(s)} \times G_M \times G_{gh}.
\end{aligned} \tag{18}$$

and try to compute in the $(p,q) = (4,3)$ case, $i.e.$,

Gravatational Ising model.

[communication with Al. Z]

Gravity part by Liouville

$$\tilde{G}_L^{(1)} = -\frac{\pi^2}{\sin\left(\frac{2\pi q}{p}\right)} |z_{13}|^{-4h_1}|z_{24}|^{-4h_2}|\xi(1-\xi)|^{-2\beta_1\beta_2}$$
$$\times \left\{ {}_2F_1\left(-A, 1+A; 1; \xi\right) {}_2F_1\left(-A, 1+A; 1; 1-\bar{\xi}\right) + (c.c.)\right\}, \tag{19}$$

where
$$A = 1 - \frac{2q}{p}, \quad \frac{\beta_1}{\alpha} = 1 - \frac{p}{2q}, \quad \frac{\beta_2}{\alpha} = -1 + \frac{p}{q},$$
$$h_1 = -\frac{3(p-2q)}{4q}, \quad h_2 = 2 - \frac{2q}{p}. \tag{20}$$

In the $(p,q) = (4,3)$ case, $O_1 = O_2$ and $\Phi_{2,1} \equiv \epsilon$, and

$$h = \frac{1}{2}, \quad \alpha = -\sqrt{\frac{3}{2}}, \quad Q = -\frac{7}{\sqrt{6}}, \quad \beta = -\frac{1}{\sqrt{6}}, \quad \beta^2 = \frac{1}{6}, \quad A = -\frac{1}{2}. \tag{21}$$

Hence,

$$\tilde{G}_L^{(1)} = \pi^2 |z_{13}z_{24}|^{-2}|\xi(1-\xi)|^{-\frac{1}{3}}$$
$$\times \left\{ {}_2F_1\left(1/2, 1/2; 1; \xi\right) {}_2F_1\left(1/2, 1/2; 1; 1-\bar{\xi}\right) + (c.c.)\right\}. \tag{22}$$

Matter part by Coulomb gas

$$\begin{aligned}
&\langle\, O_{2,1} O_{q-1,p-3} O_{2,1} O_{q-1,p-3} \rangle_M \\
&= \quad |z_{13}|^{-4h_{2,1}} |z_{24}|^{-4h_{q-1,p-3}} |\xi|^{2\alpha_1\alpha_2} |1-\xi|^{-2\alpha_1\alpha_2+2-\frac{2p}{q}}
\end{aligned}$$

$$\times \left[G_1 \left| {}_2F_1\left(\frac{p}{q}, 1-\frac{p}{q}; 4-\frac{p}{q}; \xi\right) \right|^2 + G_2 \left| \frac{\xi}{1-\xi} \right|^{-2\left(3-\frac{p}{q}\right)} \left| {}_2F_1\left(\frac{p}{q}, 1-\frac{p}{q}; -2+\frac{p}{q}; \xi\right) \right|^2 \right]$$

where G_i's are some constants which satisfy the crossing symmetry and monodoromy invariance, and

$$\begin{aligned}
c_M &= 1 - \frac{6(p-q)^2}{pq} = 1 - 12\alpha_0^2, \quad \alpha_0 = \frac{p-q}{\sqrt{2pq}}, \quad \alpha_+ = \sqrt{\frac{2p}{q}},\ \alpha_- = -\sqrt{\frac{2q}{p}}, \\
\alpha_1 &= -\frac{\alpha_+}{2}, \quad \alpha_2 = \frac{2-q}{2}\alpha_+ + \frac{4-p}{2}\alpha_-, \\
h^M_{2,1} &= \frac{3p-2q}{4q}, \quad h^M_{q-1,p-3} = -1 + \frac{2q}{p}, \quad \alpha_1\alpha_2 = p - 2 - \frac{p}{q}
\end{aligned} \tag{23}$$

In the $(p,q) = (4,3)$ case,

$$\langle\, \epsilon\epsilon\epsilon\epsilon \rangle_M = G_2\, |z_{13}z_{24}|^{-2} |\xi(1-\xi)|^{-2} \left| 1 - \xi + \xi^2 \right|^2, \tag{24}$$

A full correlation function of the Grav-Ising model

Plugging all into one, one gets a moduli integral over $\mathbf{C}$:

$$\begin{aligned} \langle\, \epsilon\epsilon\epsilon\epsilon \rangle_{All} &= \langle\, O_{2,1}O_{2,1}O_{2,1}O_{2,1} \rangle \\ &= \int \prod_{i=1}^{4} d^2 z_i \langle\, \epsilon\epsilon\epsilon\epsilon \rangle_L \langle\, \epsilon\epsilon\epsilon\epsilon \rangle_M \langle\, \text{ghost factor} \rangle \\ &= (\mu \ln \mu)\, G' \int d^2\xi\, |\xi(1-\xi)|^{-7/3} \left|1 - \xi + \xi^2\right|^2 \\ &\quad \times \left[K(\xi)\, K\left(1-\bar{\xi}\right) + K(1-\xi)\, K\left(\bar{\xi}\right) \right] \\ &= (\mu \ln \mu)\, G. \end{aligned} \tag{25}$$

According to $c = \frac{1}{2}$ matrix model, $G = 0$ [Crnković, Ginsparg, Moore '89]. We infer that G is a finite real constant. However, a straightforward calculation with no regularisation shows that this moduli integral is divergent. Because...

1. The integral is of the same order as the integral on the unit disk D.
2. The integral on D is apparently divergent, since $\int_D |\xi(1-\xi)|^{-\frac{7}{3}} \sim \infty$.

We would need some regularisation such as analytic continuation.

appearing in the calculation for the Virasoro-Shapiro amplitude...

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Meaning of Logarithms in minimal CFT

[Gurarie '93]

At $c = -2$,

$$\begin{aligned} & \langle\, \Phi_{1,2}\Phi_{1,2}\Phi_{1,2}\Phi_{1,2}\rangle \\ \sim\quad & |z_{13}z_{24}|^{1/4}|\xi(1-\xi)|^{1/4} \\ & \times \left(K(\xi)K(1-\bar{\xi}) + K(1-\xi)K(\bar{\xi})\right) \end{aligned} \tag{26}$$

Operator algebra (Fusion rules) for ordinary CFT are:

$$\Phi_1(z)\Phi_2(0) = \sum_{\{h_3\}} z^{-h_1-h_2+h_3}\Phi_3(0). \tag{27}$$

But the above implies a fusion rule:

$$\Phi_{1,2}(z)\Phi_{1,2}(0) \sim z^{1/4}\left(C(0)\ln z + D(0)\right) + \cdots . \tag{28}$$

Conformal property of $C(0)$ and $D(0)$ shows the Jordan cell under L_0:

States:

$$L_0\,|C\rangle = 0, \quad L_0\,|D\rangle = |C\rangle\,. \tag{29}$$

LCFT (Logarithmic Conformal FT)

[Gurarie '93, Knizhnik '87]

- Non-unitarity —— Unitarity
- Logarithmically Degenerate fields

Correlation functions [Caux,Kogan,Tsvelik '96]

$$\langle\, C(z)C(w)\rangle \;\sim\; 0, \quad \langle\, C(z)D(w)\rangle \sim \frac{\alpha}{(z-w)^{2h_C}},$$
$$\langle\, D(z)D(w)\rangle \;\sim\; \frac{1}{(z-w)^{2h_C}}\left(-2\alpha\ln(z-w)+\alpha'\right) . \tag{30}$$

States:

$$L_0\,|C\rangle \;=\; h_C\,|C\rangle\,,$$
$$L_0\,|D\rangle \;=\; h_C\,|D\rangle + |C\rangle\,. \tag{31}$$

Fusion rules of the Liouville theory

Structure constant, $i.e.$, 3pt func. [Dorn, Otto '92][Zamolodchikov^2 '95]

$$C_{\beta_1,\beta_2,\beta_3} = \left(\pi\mu\cdot\gamma\left(\alpha^2/2\right)\cdot\left(\alpha^2/2\right)^{1-\alpha^2/2}\right)^s \times\frac{\Upsilon(\alpha/\sqrt{2})}{\Upsilon((\beta+Q)/\sqrt{2})}\prod_{i=1}^{3}\frac{\Upsilon(\sqrt{2}\beta_i)}{\Upsilon((\beta-2\beta_i)/\sqrt{2})} \qquad (32)$$

where $\beta=\sum\beta_i$, $\gamma(x)=\Gamma(x)/\Gamma(1-x)$, and $\Upsilon(x)$ is the Barnes double gamma func., which is an entire func with zeros in the 1st and 3rd quadraple on the lattice $(m,n)\in\mathbf{Z}^2$, $i.e.$, C_{123} has poles at

$$x = x_{m,n} \equiv \left(m\alpha + n\cdot\frac{2}{\alpha}\right)/\sqrt{2}.$$

Fusion rules [Curtright, Thorn '82][...]

$$e^{\beta_1\phi}(x)e^{\beta_2\phi}(0) = \int\frac{dP}{2\pi}C^{-Q/2+iP}_{\beta_1,\beta_2}x^{P^2+\cdots}e^{\frac{1}{\sqrt{2}}(Q/2-iP)\phi}. \qquad (33)$$

Fusion rules of the Liouville theory

A pole at $P = 0$, gives something like

$$\frac{1}{\epsilon} e^{-Q/2\phi},$$

with a small $\epsilon \to 0$.

Remind you that ... [Kogan, Wheater '00][Rasmussen '04][...]

$$O_1' = \frac{1}{\epsilon}(O_1 + (1 + \epsilon^2)O_2), \quad O_2' = \epsilon O_2. \tag{34}$$

form a C-D-like states. Maybe, it may renown the interpretation of the structure constant and corresponding boundary states in the continuous spectrum...

Concluding remarks

♠ We have studied (p, q) minimal gravity (or MST).

showed the explicit form of the full correlation function with logarithms in the gravitational Ising model, in particular.

♡ No strange conditions for logarithms for generic p, q.
♡ Convergence of the moduli integration is still under investigation.

Minimal String Theory is Logarithmic!
Emergence of $\ln \mu$ – inconsistent to Matrix Models?

The logarithms in the Liouville sector seem to suggest that the fields are logarithmically degenerate in fusion rules of the Liouville sector.

Interesting directions:

— Precise structure of local continuum theory (at the zero-volume limit).
— The effects of boundary
— Relation to Branes, Rings, Matrix Models, etc.