Recent Developments on N=8 Supergravity

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1. Introduction

- **Current status of supergravity**

It is an interesting challenge to formulate gravitational interaction in a way consistent with quantum mechanics.

Pure gravity is UV finite at 1-loop, since the counter term is given by Gauss-Bonnet one.

t'Hooft, Veltman

$$G_N^2(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$$ ~ **total derivative**

At 2-loop, however, there appear $G_N^3 R^3$ counter terms, which are UV divergent.

Supersymmetry has been studied to remove UV divergence.
Maximal supergravity is N=8 in D=4.

32 super charges

Is N=8 supergravity UV finite?

At least there are R^4 terms at 3-loop.

- **UV divergence in N=8 supergravity**

From 2-loop calculation, it was speculated on the critical dimension which contains UV divergence at L-loop.

$$D_c = \frac{10}{L} + 2$$

Bern et al.(1998)

Explicit 3-loop calculation, however, tells us that $D_c = 6$ for $L = 3$. This critical dimension is the same as that of N=4 super Yang-Mills theory.

Bern et al.(2007)

$$D_c = \frac{6}{L} + 4 \; ?$$

If this is true, we have D=4 UV finite point particle theory of gravity.

- ## How to calculate?

Three key techniques:

1. Perturbative QCD calculation (spinor helicity formalism)
2. Cut construction method

$$A^{\text{1-loop}} \sim A^{\text{tree}} \times A^{\text{tree}} \sim I(s,t,u)A^{\text{tree}}$$

3. Kawai-Lewellen-Tye relation (KLT)

$$\underset{\text{gravity}}{M^{\text{tree}}} \sim \underset{\text{YM}}{A^{\text{tree}}} \times \underset{\text{YM}}{A^{\text{tree}}}$$

By using these, we can calculate loop amplitudes of gravity.

$$\begin{aligned} M^{\text{1-loop}} &\sim M^{\text{tree}} \times M^{\text{tree}} \\ &\sim (A^{\text{tree}})^4 \\ &\sim I'(s,t,u)M^{\text{tree}} \end{aligned}$$

Plan :

1. Introduction
2. Spinor Helicity Formalism and Cut Construction Method
3. KLT Relation at 3 and 4 Points Interactions
4. Loop Calculation and Finiteness in N=8 SUGRA
5. Non-renormalization Conditions in Type IIA
6. Higher Derivative Corrections in String Theory
7. 11 dim. Supergravity
8. Higher Derivative Corrections in M-Theory
9. Summary

2. Spinor Helicity Formalism and Cut Construction Method

First let us calculate 4-point gluon amplitude in N=4 SYM by employing spinor helicity formalism.

In this formalism, color factors are separated, and gluon polarization vector is expressed by spinors.

$$\epsilon_\mu^+(k,q) = \frac{\langle q^-|\gamma_\mu|k^-\rangle}{\sqrt{2}\langle qk\rangle}, \quad \epsilon_\mu^-(k,q) = -\frac{\langle q^+|\gamma_\mu|k^+\rangle}{\sqrt{2}[qk]}$$

Notations are as follows:

$|k_i^\pm\rangle = |i^\pm\rangle = u_\pm(k_i)$: massless on shell spinor with helicity $\pm$

$$\langle qk\rangle \equiv \overline{u_-(q)}u_+(k)$$

$$[qk] \equiv \overline{u_+(q)}u_-(k)$$

$$\epsilon_i^{\pm\mu}(q) \equiv \epsilon^{\pm\mu}(k_i,q)$$

q is called a reference momentum and can be chosen arbitrarily.

Gauge transformation is realized by a shift of reference momentum.

$$\epsilon_\mu^+(\tilde{q}) - \epsilon_\mu^+(q) = -\frac{\sqrt{2}\langle\tilde{q}q\rangle}{\langle\tilde{q}k\rangle\langle qk\rangle} \times k_\mu$$

Following relations are useful and make calculations simple.

$$\epsilon_i^\pm(q)\cdot q = 0, \qquad \epsilon_i^{\pm\mu}(q) \equiv \epsilon^{\pm\mu}(k_i, q)$$

$$\epsilon_i^+(q)\cdot\epsilon_j^+(q) = \epsilon_i^-(q)\cdot\epsilon_j^-(q) = 0$$

$$\epsilon_i^+(k_j)\cdot\epsilon_j^-(q) = \epsilon_i^+(q)\cdot\epsilon_j^-(k_i) = 0$$

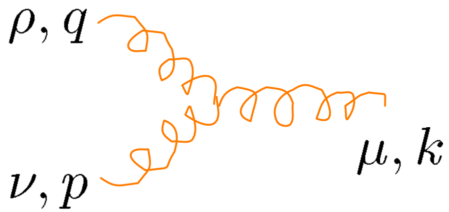
From these,

$$A_n^{\text{tree}}(1^\pm, 2^+, \cdots, n^+) = 0$$

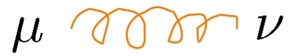
Maximally Helicity Violating (MHV) amplitude is

$$A_n^{\text{tree}}(1^-, 2^-, 3^+, \cdots, n^+) \neq 0$$

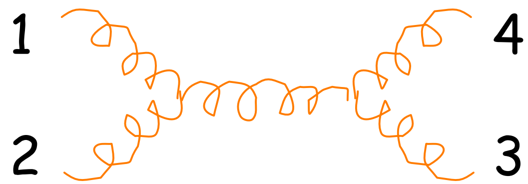
- **Tree level 4 gluon MHV amplitude**



$$\frac{i}{\sqrt{2}}(\eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu + \eta_{\mu\nu}(k-p)_\rho)$$



$$-i\eta_{\mu\nu}\frac{1}{p^2}$$



$$s=(k_1+k_2)^2, t=(k_2+k_3)^2, u=(k_1+k_3)^2$$

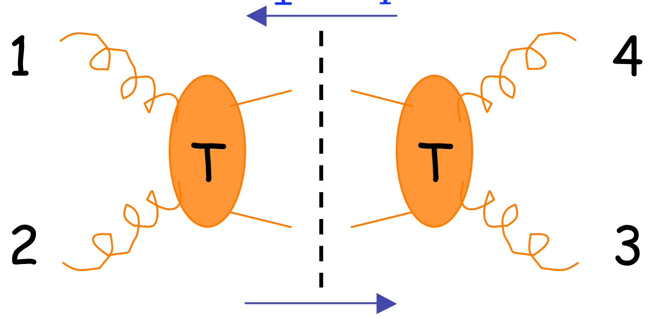
$$\begin{aligned}
&A_4^{\text{tree}}(1^-,2^-,3^+,4^+)\\
&=\left(\frac{i}{\sqrt{2}}\right)^2\left(\frac{-i}{s}\right)[\epsilon_1^-\cdot\epsilon_2^-(k_1-k_2)^\mu+\epsilon_2^{-\mu}\epsilon_1^-\cdot(2k_2+k_1)+\epsilon_1^{-\mu}\epsilon_2^-\cdot(-2k_1-k_2)]\\
&\qquad\times[\epsilon_3^+\cdot\epsilon_4^+(k_3-k_4)_\mu+\epsilon_{4\mu}^+\epsilon_3^+\cdot(2k_4+k_3)+\epsilon_{3\mu}^+\epsilon_4^+\cdot(-2k_3-k_4)]\\
&=-\frac{2i}{s}(\epsilon_2^-\cdot\epsilon_3^+)(\epsilon_1^-\cdot k_2)(\epsilon_4^+\cdot k_3)\\
&=i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}
\end{aligned}$$

• 1-Loop 4 gluon amplitude by cut construction method

Loop amplitudes can be constructed via unitarity.

$$1 = S^\dagger S = (1 - iT^\dagger)(1 + iT^\dagger) \;\Rightarrow\; 2\mathrm{Im}T = T^\dagger T$$

$\ell_1 = p$



$\ell_2 = p - k_1 - k_2$

$$A_4^{\text{1-loop}}(1^-, 2^-, 3^+, 4^+)\Big|_{\text{s-cut}}$$
$$= \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{i}{\ell_1^2} A_4^{\text{tree}}(-\ell_1^+, 1^-, 2^-, \ell_2^+) \frac{i}{\ell_2^2} A_4^{\text{tree}}(-\ell_2^-, 3^+, 4^+, \ell_1^-)\Big|_{\text{s-cut}}$$

A product of tree level amplitudes is given by

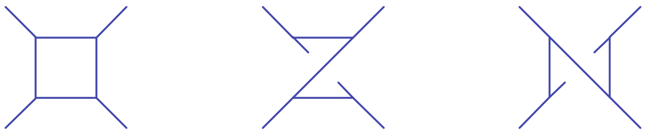
$$
\begin{aligned}
& A_4^{\text{tree}}(-\ell_1^+, 1^-, 2^-, \ell_2^+) A_4^{\text{tree}}(-\ell_2^-, 3^+, 4^+, \ell_1^-) \\
&= -\frac{\langle 12\rangle^4 \langle \ell_1 - \ell_2\rangle^4}{\langle 12\rangle\langle 2\ell_2\rangle\langle \ell_2 - \ell_1\rangle\langle -\ell_1 1\rangle\langle \ell_1 - \ell_2\rangle\langle -\ell_2 3\rangle\langle 34\rangle\langle 4\ell_1\rangle} \\
&= \frac{i\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \frac{-i\langle 23\rangle\langle 41\rangle\langle \ell_1 - \ell_2\rangle^2}{\langle 2\ell_2\rangle\langle -\ell_1 1\rangle\langle 4\ell_1\rangle\langle -\ell_2 3\rangle} \\
&= i A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+) \frac{[\ell_1 1]\langle 14\rangle[4\ell_1]\langle \ell_1\ell_2\rangle[\ell_2 3]\langle 32\rangle[2\ell_2]\langle \ell_2\ell_1\rangle}{(p-k_1)^4 (p+k_4)^4} \\
&= i A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+) \frac{\text{tr}(\ell_1 1 4 \ell_1 \ell_2 3 2 \ell_2)}{(p-k_1)^4 (p+k_4)^4} \\
&= -ist A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+) \frac{1}{(p-k_1)^2 (p+k_4)^2}
\end{aligned}
$$

1-loop 4 gluon amplitude is given by

$$
\begin{aligned}
&A_4^{\text{1-loop}}(1^-,2^-,3^+,4^+)\Big|_{\text{s-cut}} \\
&= istA_4^{\text{tree}}(1^-,2^-,3^+,4^+)\int\frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}}\frac{1}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+k_4)^2}\Big|_{\text{s-cut}} \\
&= istA_4^{\text{tree}}(1^-,2^-,3^+,4^+)\mathcal{I}_4^1(s,t)\Big|_{\text{s-cut}}
\end{aligned}
$$

Combining t-channel cut, the amplitude becomes

$$
\begin{aligned}
&A_4^{\text{1-loop}}(1^-,2^-,3^+,4^+) \\
&= istA_4^{\text{tree}}(1^-,2^-,3^+,4^+)\Big\{\mathcal{I}_4^1(s,t)+\mathcal{I}_4^1(t,u)+\mathcal{I}_4^1(u,s)\Big\}
\end{aligned}
$$



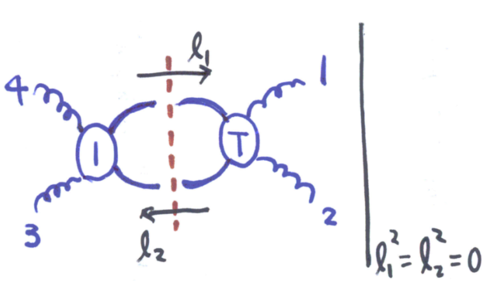
• 2-loop 4pt in N=4 SYM

Bern, Rozowsky, Yan

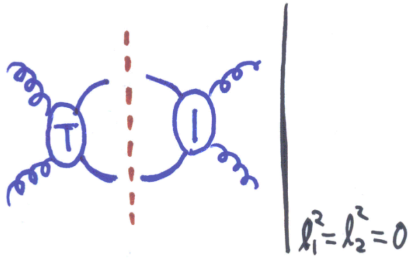
Bern, Dixon, Dunbar, Perelstein, Rozowsky

2 loop amplitude is constructed by cutting tree and 1-loop.

$$A_4^{\text{SYM 2-loop}} =$$

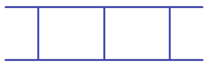
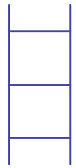


$+$



$+ \cdots$

$$= istA_4^{\text{tree}}(is)I_4^P(s,t) + istA_4^{\text{tree}}(it)I_4^P(t,s) + \cdots$$

is  **Rung rule** it 

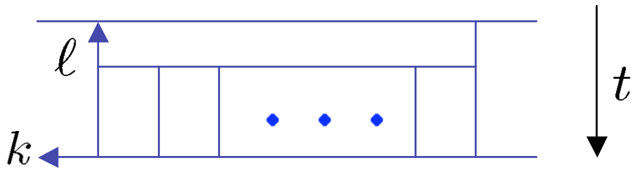
$$\rightarrow \frac{2d}{14}$$

UV divergence arises in 7 dimensions

$$t_8 D^2 F^4$$

- **Estimation of UV divergences by rung rule**

Let us evaluate L-loop diagram by using rung rule



$$A_4^{\text{L-loop}} \sim st A_4^{\text{tree}} t \int d^{dL}\ell \frac{(\ell+k)^{2(L-2)}}{(\ell^2)^{3L+1}}$$

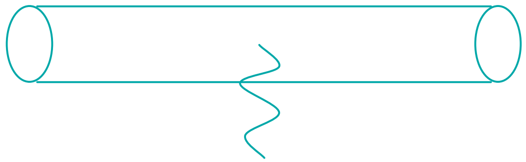
$$\longrightarrow \frac{(d+2)L-4}{6L+2}$$

Critical dimension for N=4 SYM is given by

$$D_c^{\text{YM}} = 4 + \frac{6}{L}$$

3. KLT Relation at 3 and 4 points Interactions

Here we see that 3pt interaction of gravitons in supergravity is really obtained by 3pt interaction of superstrings.



$$
\begin{aligned}
M_3 &= \langle \zeta_1 \otimes \tilde{\zeta}_1, k_1 | V_O(\zeta_2, \tfrac{1}{2}k_2) V_O(\tilde{\zeta}_2, \tfrac{1}{2}k_2) | \zeta_3 \otimes \tilde{\zeta}_3, k_3 \rangle \\
&= A_3 \otimes \tilde{A}_3
\end{aligned}
$$

3pt amplitude in closed string theory is equal to the tensor product of two 3pt amplitudes in open string theory. (KLT relation)

- **3 gluon amplitude**

$$A_3 = g\,\langle \zeta_1, k_1 | V_O(\zeta_2, k_2) | \zeta_3, k_3 \rangle$$

$$\sim g \langle \zeta_1 | \Big(\zeta_2 \cdot k_3 - \zeta_2^i k_2^j \underline{R_0^{ij}} \Big) | \zeta_3 \rangle$$

Generator of rotation in (x^1,…,x^8)

$$\langle \zeta_1 | R_0^{ij} | \zeta_3 \rangle = \zeta_1^i \zeta_3^j - \zeta_1^j \zeta_3^i$$

$$\sim g\Big((\zeta_1 \cdot k_2)(\zeta_2 \cdot \zeta_3) + (\zeta_2 \cdot k_3)(\zeta_3 \cdot \zeta_1) + (\zeta_3 \cdot k_1)(\zeta_1 \cdot \zeta_2) \Big)$$

$$k^2 = k \cdot \zeta = \sum_{i=1}^{3} k_i = 0$$

When group factors are included, this expression corresponds to the interaction term in Yang-Mills theory.

$$A_3 \sim \mathrm{tr}\Big((\partial_\mu A_\nu - \partial_\nu A_\mu)[A^\mu, A^\nu] \Big)$$

- **3 graviton amplitude**

$$M_3 = g^2\Big((\zeta_1\cdot\zeta_2)(\zeta_3\cdot k_1) + (\zeta_1\cdot\zeta_3)(\zeta_2\cdot k_3) + (\zeta_2\cdot\zeta_3)(\zeta_1\cdot k_2)\Big)$$

$$\otimes\Big((\tilde\zeta_1\cdot\tilde\zeta_2)(\tilde\zeta_3\cdot k_1) + (\tilde\zeta_1\cdot\tilde\zeta_3)(\tilde\zeta_2\cdot k_3) + (\tilde\zeta_2\cdot\tilde\zeta_3)(\tilde\zeta_1\cdot k_2)\Big)$$

$$\sim 3g^2(h^{\alpha\beta}h^{\gamma\delta}\partial_\alpha\partial_\beta h_{\gamma\delta} + 2h^{\alpha\beta}\partial_\gamma h_{\delta\alpha}\partial_\beta h^{\gamma\delta})$$

$$k^2 = k\cdot\zeta = k\cdot\tilde\zeta = \sum_{i=1}^{3} k_i = 0$$

In the last line, graviton $h_{\alpha\beta}$ is identified with symmetric traceless part of $\zeta_\alpha\otimes\tilde\zeta_\beta$

$(b_{\alpha\beta}\sim$ antisymmetric part of $\zeta_\alpha\otimes\tilde\zeta_\beta)$

$(\phi\sim$ trace part of $\zeta_\alpha\otimes\tilde\zeta_\beta)$

This correctly reproduce 3 point interaction in Einstein-Hilbert action.

$$R\sim\gamma_{\alpha\beta\gamma}\gamma^{\alpha\beta\gamma} - \frac{1}{4}\Big(h^{\alpha\beta}h^{\gamma\delta}\partial_\alpha\partial_\beta h_{\gamma\delta} + 2h^{\alpha\beta}\partial_\gamma h_{\delta\alpha}\partial_\beta h^{\gamma\delta}\Big)$$

- **KLT Relation (Tree level)**

Kawai, Lewellen, Tye

In the low energy limit :

Open string^2 ⟷ Closed string

↓ ↓

Gauge Theory Gravity

$$\left(\mathrm{tr}\left((\partial_\mu A_\nu - \partial_\nu A_\mu)[A^\mu, A^\nu]\right)\right)^2 \longleftrightarrow \sqrt{-g}R$$

Since we are considering maximally supersymmetric case, the above statement becomes

(Super Yang-Mills)^2 ⟷ Supergravity

N=4 N=8

• Tree level 4-point amplitudes

Tree level 4 graviton amplitude is expressed as a "product" of two tree level 4 point gluon amplitudes.

$$A_4^{\text{op tree}} \sim \frac{\Gamma(-s/2)\Gamma(-t/2)}{\Gamma(1+u/2)} t_8 F^4$$

$$M_4^{\text{cl tree}} \sim -\pi \frac{\Gamma(-s/8)\Gamma(-t/8)\Gamma(-u/8)}{\Gamma(1+s/8)\Gamma(1+t/8)\Gamma(1+u/8)} t_8 F^4 \otimes t_8 \tilde{F}^4$$

By using the relation of gamma functions,

$$-\pi \frac{\Gamma(-s/8)\Gamma(-t/8)\Gamma(-u/8)}{\Gamma(1+s/8)\Gamma(1+t/8)\Gamma(1+u/8)} = \sin\left(\frac{\pi t}{8}\right) \frac{\Gamma(-s/8)\Gamma(-t/8)}{\Gamma(1+u/8)} \frac{\Gamma(-t/8)\Gamma(-u/8)}{\Gamma(1+s/8)}$$

we obtain KLT relation (closed)~(open)^2

$$M_4^{\text{cl tree}} \sim \sin\left(\frac{\pi t}{8}\right) A_4^{\text{op tree}} \times A_4^{\text{op tree}}$$

$$istu M_4^{\text{tree}} \sim st A_4^{\text{tree}} \times tu A_4^{\text{tree}}$$ **Low energy**

KLT relation makes it possible to calculate gravity amplitudes by recycling YM amplitudes

4. Loop Calculation and Finiteness in N=8 SUGRA

- **Comparison between YM tree and SUGRA tree**

Recall tree level 4pt amplitudes in N=4 SYM.

$$A_4^{\text{tree}}(1^-,2^-,3^+,4^+) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} = -\frac{i}{t}\times\left[s\frac{\langle 12\rangle[34]}{[12]\langle 34\rangle}\right] = -\frac{i}{t}C$$

$$A_4^{\text{tree}}(1^-,2^-,4^+,3^+) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 31\rangle} = -\frac{i}{u}\times\left[s\frac{\langle 12\rangle[34]}{[12]\langle 34\rangle}\right] = -\frac{i}{u}C$$

By using these tree level 4pt amplitude in N=8 SUGRA becomes

$$\begin{aligned} M_4^{\text{tree}}(1^-,2^-,3^+,4^+) &= -isA_4^{\text{tree}}(1^-,2^-,3^+,4^+)A_4^{\text{tree}}(1^-,2^-,4^+,3^+) \\ &= -i\left(\frac{1}{t}+\frac{1}{u}\right)\times\left[s\frac{\langle 12\rangle[34]}{[12]\langle 34\rangle}\right]^2 \\ &= -i\left(\frac{1}{t}+\frac{1}{u}\right)C^2 \end{aligned}$$

The coefficients for graviton amplitudes are squares of the coefficients for gluon amplitudes.

• 1-loop 4pt amplitudes in N=8 SUGRA

A product of 2 tree level amplitudes is evaluated as

$$
\begin{aligned}
& M_4^{\text{tree}}(-\ell_1^+,1^-,2^-,\ell_2^+)M_4^{\text{tree}}(-\ell_2^-,3^+,4^+,\ell_1^-) \\
= & -s^2 A_4^{\text{tree}}(-\ell_1^+,1^-,2^-,\ell_2^+)A_4^{\text{tree}}(-\ell_2^-,3^+,4^+,\ell_1^-)A_4^{\text{tree}}(\ell_2^+,1^-,2^-,-\ell_1^+)A_4^{\text{tree}}(\ell_1^-,3^+,4^+,-\ell_2^-) \\
= & \left\{stA_4^{\text{tree}}(1^-,2^-,3^+,4^+)\right\}^2 \frac{s^2}{(\ell_1-k_1)^2(\ell_2-k_3)^2(\ell_2+k_1)^2(\ell_1+k_3)^2} \\
= & istuM_4^{\text{tree}}(1,2,3,4)\left(\frac{1}{(\ell_1-k_1)^2}+\frac{1}{(\ell_1-k_2)^2}\right)\left(\frac{1}{(\ell_2-k_3)^2}+\frac{1}{(\ell_2-k_4)^2}\right)
\end{aligned}
$$

Combining the above expression with the cut construction method, 1-loop 4pt amplitudes are obtained.

$$
\begin{aligned}
M_4^{\text{1-loop}}(1^-,2^-,3^+,4^+) &= \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{i}{\ell_1^2} M_4^{\text{tree}}(-\ell_1^+,1^-,2^-,\ell_2^+)\frac{i}{\ell_2^2}M_4^{\text{tree}}(-\ell_2^-,3^+,4^+,\ell_1^-) \\
&= istuM_4^{\text{tree}}(1,2,3,4)\left\{\mathcal{I}_4^1(s,t)+\mathcal{I}_4^1(t,u)+\mathcal{I}_4^1(u,s)\right\}
\end{aligned}
$$

Green, Schwarz, Brink

Again the coefficients for graviton amplitudes are squares of the coefficients for gluon amplitudes.

- **2-loop 4pt in N=8 SUGRA**

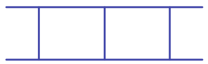
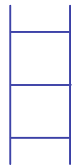
Bern, Rozowsky, Yan

Bern, Dixon, Dunbar, Perelstein, Rozowsky

2-loop 4pt amplitudes for N=8 SUGRA can be obtained by squaring the SYM factor

$$M_4^{\text{2-loop}} = (stA_4^{\text{tree}})^2(s^2 I_4^P(s,t) + t^2 I_4^P(t,s) + \cdots)$$

$$= stuM_4^{\text{tree}}(\underline{is^2 I_4^P(s,t)} + \underline{it^2 I_4^P(t,s)} + \cdots)$$

is^2  it^2 

Rung rule

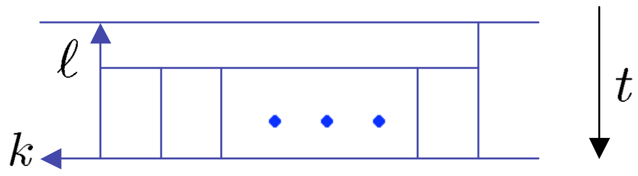
$\longrightarrow \frac{2d}{14}$

UV divergence arises in 7 dimensions

$$t_8 t_8 D^4 R^4$$

• Estimation of UV divergences in N=8 SUGRA by rung rule

Let us evaluate L-loop diagram



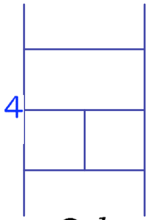
$$M_4^{\text{L-loop}} \sim stu M_4^{\text{tree}} t^2 \int d^{dL}\ell \frac{(\ell+k)^{4(L-2)}}{(\ell^2)^{3L+1}} \qquad \frac{(d+4)L-8}{6L+2}$$

Critical dimension for N=8 SUGRA is given by

$$D_c^{SUGRA} = 2 + \frac{10}{L}$$

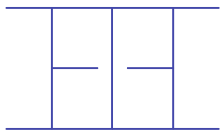
- **3-loop calculation tells ...**

From 2 particle cuts construction, we obtain

s^4  $t^2(\ell+k)^4$  ...

$\frac{3d}{20}$ $\frac{3d}{16}$

Leading divergence in the right diagram can be cancelled by non-rung-rule diagram



$\frac{3d}{16} \longrightarrow \frac{3d}{18}$

4pt graviton amplitude at 3-loop is divergent in 6 dimension. This is the same as N=4 SYM. Thus the conjecture is

$$D_c^{SUGRA} = 4 + \frac{6}{L} \qquad \text{for N=8 SUGRA.}$$

If this is true, D=4 N=8 SUGRA is UV finite.

5. Non-renormalization Conditions in Type IIA

- **11 dim. L-loop amplitude on a torus** Green, Russo, Vanhove

Calculation of 11 dim. L-loop amplitude is difficult. By using power counting, however, we can restrict its form. The L-loop amplitude on a torus which includes $D^{2k}R^4$ term will be

$$S_L = \sum_{w=0}^{w_L} \underbrace{\ell_p^{9(L-1)}}_{-9L+9} \Lambda^{\boxed{9L-6-2\beta_L}} \underbrace{(\Lambda V^{\frac{1}{2}})^{-w}}_{\text{subdivergences}} \underbrace{\int d^9x\sqrt{-g}V}_{-11} f(\Omega,\bar{\Omega}) \underbrace{(VD^2)^v}_{\text{derivatives}} \underbrace{D^{2\beta_L}R^4}_{2\beta_L+8}$$

$$= \sum_q \ell_p^{9(L-1)} \Lambda^{9L-6-2k-2q} \int d^9x\sqrt{-g}V^{1-q} f(\Omega,\bar{\Omega}) D^{2k}R^4$$

where

$$k = v + \beta_L, \quad q = \tfrac{1}{2}w - v, \quad V = R_9 R_{11}$$

$$f(\Omega,\bar{\Omega}) = \Omega_2^a + \cdots + \Omega_2^{a-2h} + \cdots + (\text{Euclidean D0})$$

And following constraints are imposed

$$\beta_1 = 0,\ \beta_2 = 2,\ \beta_L \geq 2(L>2) \qquad w \geq 0$$

• Higher derivative action in type IIA

By considering the duality between M and IIA,

$$\ell_p = \ell_s g_s, \quad R_{11} = \ell_s g_s$$

$$\hat{\Lambda} = \Lambda \ell_p, \quad r_A = R_9/\ell_s, \quad \Omega_2 = R_9/R_{11}$$

Effective action in IIA from genus h can be derived

$$S_L^{(k)} = \sum_q \ell_s^{2k-1} \hat{\Lambda}^{9L-6-2k-2q} \int d^9x \sqrt{-g}\, r_A^{3-\frac{4}{3}q+\frac{2}{3}k} (\cdots + r_A^{-2h} e^{2(h-1)\phi} + \cdots) D^{2k} R^4$$

$$a = 2 + \tfrac{2}{3}k - \tfrac{1}{3}q$$

Terms which are linear to r_A survive after decompactification

$$2q = 3 + k - 3h \qquad \text{or} \qquad w = 3 + 3k - 3h - 2\beta_L$$

$w \geq 0$ gives $h \leq k + 1 - \frac{2}{3}\beta_L$

$$L = 1 \qquad h - 1 \leq k$$

$$L > 1 \qquad h < k$$

Then we obtain strong conditions.

• Conditions for higher derivative terms in type IIA

h-loop amplitude in type IIA contributes as follows.

1. No contributions to $R^4, D^2R^4, \cdots, D^{2(h-1)}R^4$
2. $D^{2h}R^4$ can be determined by 1-loop in 11 dim.
3. $D^{2(h+1)}, \cdots$ are permitted and may arise from L-loop in 11 dim. (L>1)

Leading term in low energy can be given by

$$S_{(10)} = \underbrace{\ell_s^{8(h-1)}}_{-8h+8} \Lambda_{10}^{\boxed{6h-6}} \int \underbrace{d^{10}x}_{-10} \sqrt{-g} e^{2(h-1)\phi} \underbrace{D^{2h}R^4}_{2h+8}$$

- **Dimensional reduction**

If the same property holds for lower dimensions, the leading contributions can be given by

$$S_{(d)} = \ell_s^{(d-2)(h-1)} \Lambda_d^{(d-4)h-6} \int d^d x \sqrt{-g} e^{2(h-1)\phi} D^{2h} R^4$$

If this is true, UV divergences are absent in dimensions which satisfy

$$d < 4 + \frac{6}{h}$$

4 dim. N=8 supergravity seems to be UV finite

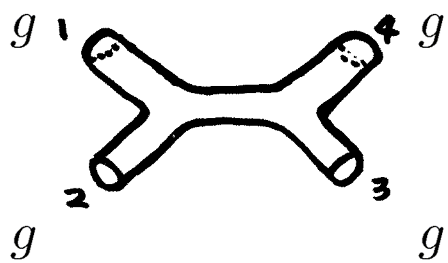
6. Higher Derivative Corrections in String Theory

Higher derivative corrections in string theories are considerably investigated in various ways

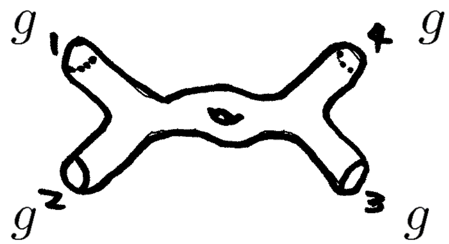
- String scattering amplitude
- Non linear sigma model
- Superfield method
- Duality
- Noether's method … and so on

For example, corrections which include R^4 terms are obtained by evaluating scattering amplitudes of 4 external gravitons

Tree



One loop



So far the complete form of the eight derivative action is not known. Known bosonic part (R^4 part) is as follows.

$$\mathcal{L}_{(\alpha')^3} \sim e^{-2\phi}\Big(t_8 t_8 e R^4 + \tfrac{1}{4\cdot 2!}\epsilon_{10}\epsilon_{10} e R^4\Big) \qquad \text{tree}$$

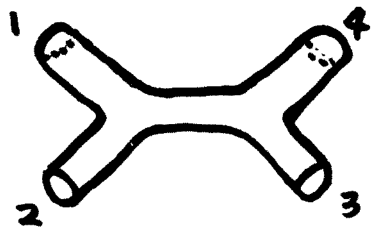
$$+\, c\Big(t_8 t_8 e R^4 - \tfrac{1}{4\cdot 2!}\epsilon_{10}\epsilon_{10} e R^4 - \tfrac{1}{6}\epsilon_{10} t_8 B R^4\Big) \qquad \text{loop}$$

$t_8 t_8 R^4$

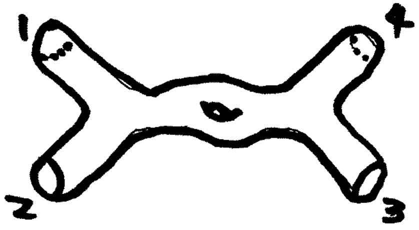
This expression is obtained by calculating the on-shell 4 graviton amplitude.

Green, Schwarz

Tree level



1-loop



$$A_4^{\text{tree}} = g^{-2} \langle V_1 | V_2 \Delta V_3 | V_4 \rangle = -g^{-2} K T$$

$$A_4^{\text{1-loop}} = \int dp \, \text{Tr}(\Delta V_1 \Delta V_2 \Delta V_3 \Delta V_4) = K I$$

Δ : propagator, V_n : massless vertex operator

$$K = t_8^{ijklmnop} t_8^{abcdefgh} f^1_{ij} f^2_{kl} f^3_{mn} f^4_{op} \otimes \tilde{f}^1_{ab} \tilde{f}^2_{cd} \tilde{f}^3_{ef} \tilde{g}^4_{gh} \qquad f_{ij} \equiv \frac{1}{2}(k_i \zeta_j - k_j \zeta_i)$$

$$= t_8^{ijklmnop} t_8^{abcdefgh} R_{ijab} \cdots R_{opgh} \qquad f_{ij} \otimes \tilde{f}_{ab} = R_{ijab}$$

Gross, Witten

Gross, Sloan

$$T = \frac{\Gamma(-s/8)\Gamma(-t/8)\Gamma(-u/8)}{\Gamma(1+s/8)\Gamma(1+t/8)\Gamma(1+u/8)} \sim -\frac{2^9}{stu} - 2\zeta(3) + \cdots$$

sugra pole

$$I = \int \frac{d^2\tau}{\tau_2^2} F(\tau, \bar{\tau}) \sim \frac{\pi}{3} + \cdots$$

$\epsilon_{10}\epsilon_{10}R^4$

This expression is obtained by calculating the beta functions of non-linear sigma model at 4 loop order.

$$\delta^{[ijklmnop]}_{[abcdefgh]} R^{ab}{}_{ij} R^{cd}{}_{kl} R^{ef}{}_{mn} R^{gh}{}_{op}$$

Grisaru, Zanon

$B \wedge \mathrm{tr}(R \wedge R) \wedge \mathrm{tr}(R \wedge R)$

In heterotic SUGRA, the 3-form field strength is defined as

$$H = dB - \alpha' \mathrm{tr}\left(\omega d\omega + \tfrac{2}{3}\omega^3\right)$$

$$dH = -\alpha' \mathrm{tr}(R \wedge R)$$

Bianchi id.

By the string-string duality between hetero/T^4 and IIA/K3, $d(*e^{-2\Phi} H_{\mathrm{IIA}}) = -\alpha' \mathrm{tr}(R \wedge R)$

10 dim. origin of this equation is given by

Vafa, Witten

$$B \wedge \mathrm{tr}(R \wedge R) \wedge \underline{\mathrm{tr}(R \wedge R)}$$

const. on K3

Again, known bosonic part (R^4 part) in type IIA is written as

$$\mathcal{L}_{(\alpha')^3} \sim e^{-2\phi} I_{\text{tree}} + c\, I_{\text{1-loop}},$$
$$I_{\text{tree}} = t_8 t_8 e R^4 + \frac{1}{4\cdot 2!}\epsilon_{10}\epsilon_{10} e R^4,$$
$$I_{\text{1-loop}} = t_8 t_8 e R^4 - \frac{1}{4\cdot 2!}\epsilon_{10}\epsilon_{10} e R^4 - \frac{1}{6}\epsilon_{10} t_8 B R^4$$

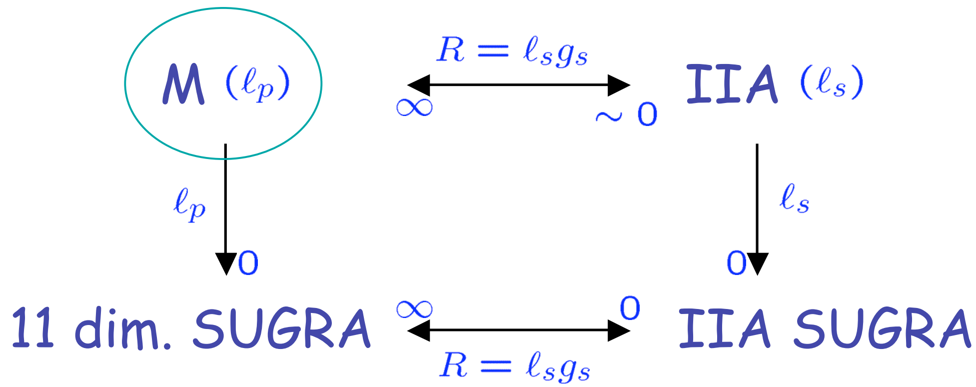
We can obtain a part of the higher derivative corrections to the N=1, D=11 supergravity by lifting the type IIA result. Then there are two candidates which will be invariant under the local supersymmetry

$$t_8 t_8 e R^4 + \frac{1}{4!}\epsilon_{11}\epsilon_{11} e R^4,$$
$$t_8 t_8 e R^4 - \frac{1}{4!}\epsilon_{11}\epsilon_{11} e R^4 - \frac{1}{6}\epsilon_{11} t_8 A R^4$$

Our goal is to check these forms by local supersymmetry

7. 11 dim. Supergravity

M-theory is defined as a strong coupling limit of type IIA superstring theory. Low energy limit is described by 11 dim. supergravity.



11 dim. supergravity consists of a vielbein, a Majorana gravitino and 3-form field

$$256 = 44 \oplus 128 \oplus 84$$
$$e^a{}_\mu, \quad \psi_\mu, \quad A_{\mu\nu\rho}$$

11 dim. SUGRA action is given by

Cremmer et al.

$$2\kappa_{11}^2 S = \int d^{11}x\, e\left(R - \frac{1}{2}\bar{\psi}_\rho \gamma^{\rho\mu\nu}\psi_{\mu\nu} - \frac{1}{2\cdot 4!}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}\right) - \frac{1}{3!}\int A\wedge F\wedge F + \cdots$$

Here the field strength of the Majorana gravitino is defined by using modified covariant derivative

$$\mathcal{D}_\mu\psi_\nu = D_\mu\psi_\nu + F_\mu\psi_\nu, \quad F_\mu = -\frac{1}{36}F_{\mu ijk}\gamma^{ijk} + \frac{1}{288}F_{ijkl}\gamma_\mu{}^{ijkl}$$

$$\psi_{\mu\nu} = \mathcal{D}_\mu\psi_\nu - \mathcal{D}_\nu\psi_\mu$$

This action possesses N=1 local supersymmetry in 11 dimensions. The susy variations are given by

$$\delta e^a{}_\mu = \bar{\epsilon}\gamma^a\psi_\mu, \quad \delta\psi_\mu = 2\mathcal{D}_\mu\epsilon, \quad \delta A_{\mu\nu\rho} = -3\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]}$$

In addition to the local supersymmetry, 11 dim. SUGRA action preserve the parity invariance.

1. General coordinate inv. and local Lorentz inv.
2. Gauge symmetry $A \to A + d\Lambda$
3. N=1 local supersymmetry
4. Parity inv. $x^{10} \to -x^{10}, A \to -A, \psi \to \gamma^{10}\psi$

M-theory should also be invariant under the above symmetries, since these are compatible with dualities.

8. Higher Derivative Corrections in M-theory

Since perturbative methods in M-theory are not developed, it is impossible to obtain higher derivative terms by evaluating scattering amplitudes of membranes. The best way to determine this structure is to use the invariance under local supersymmetry

Noether method + computer programming

Here we mainly concentrate on bosonic terms and consider cancellation of O(1) and O(F) terms step by step.

Variation	O(1)	O(F)	O(F^2)	…
$O(\psi)$	○	○		
$O(\psi^3)$	×	×	×	

Q. How many R^4 terms ?

For example let us consider a term

$$B_1[1] = R_{abcd}R_{abcd}R_{efgh}R_{efgh},$$

It is possible to assign a matrix by counting number of overlapping indices

$$\begin{pmatrix} 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix}$$

Inversely the above matrix will generate R^4 term uniquely

Thus it is important to classify possible matrices. There are 4 matrices up to permutations of 4 Riemann tensors.

$$\begin{pmatrix} 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

Finally **7 independent terms** are assigned for these matrices

$$B_1[1] = R_{abcd}R_{abcd}R_{efgh}R_{efgh}, \quad B_1[2] = R_{abcd}R_{bcdh}R_{efgh}R_{aefg},$$
$$B_1[3] = R_{abcd}R_{efgh}R_{abef}R_{cdgh}, \quad B_1[4] = R_{acbd}R_{egfh}R_{aebf}R_{cgdh},$$
$$B_1[5] = R_{abcd}R_{aefg}R_{befh}R_{cdgh}, \quad B_1[6] = R_{abcd}R_{aefg}R_{bfeh}R_{cdgh},$$
$$B_1[7] = R_{acbd}R_{aefg}R_{befh}R_{cgdh}$$

Q. How many A R^4 terms ?

There are two terms

$$B_{11}[1] = -\frac{1}{3!}A \wedge \mathrm{tr}R^2 \wedge \mathrm{tr}R^2, \quad B_{11}[2] = -\frac{1}{3!}A \wedge \mathrm{tr}R^4$$

These are related to the anomaly cancellation terms in hetero

In order to cancel variations of these bosonic terms, it is necessary to add fermionic terms to the ansatz

$$F_1 = [eR^3\bar{\psi}\psi_2]_{92}, \quad F_2 = [eR^2\bar{\psi}_2 D\psi_2]_{25}$$

Variations of the ansatz are expanded by the following terms

$$V_1 = [eR^4\bar{\epsilon}\psi]_{116}, \quad V_2 = [eR^2DR\bar{\epsilon}\psi_2]_{88}, \quad V_3 = [eR^3\bar{\epsilon}D\psi_2]_{40}$$

The cancellation mechanism up to O(F) is sketched as

$$\begin{aligned}
\delta B_1 &\sim V_1 \oplus V_2 \\
\delta B_{11} &\sim V_1 \\
\delta F_1 &\sim V_1 \oplus V_2 \oplus V_3 \\
\delta F_2 &\sim \phantom{V_1 \oplus {}} V_2 \oplus V_3
\end{aligned}$$

244 Equations among 126 Variables

After miraculous cancellation, the bosonic part is determined as

$$\mathcal{L}_{\text{boson}} = \frac{1}{24 \cdot 3^2} a \Big(t_8 t_8 R^4 + \frac{1}{4!} \epsilon_{11} \epsilon_{11} R^4 \Big) + \frac{1}{24} b \Big(t_8 t_8 R^4 - \frac{1}{4!} \epsilon_{11} \epsilon_{11} R^4 - \frac{1}{6} \epsilon_{11} t_8 A R^4 \Big)$$

The first term corresponds to tree level and the second does to one-loop part in type IIA superstring

The next step is to examine the invariance under local supersymmetry up to O(F^2)

In order to execute the cancellation to this order, we have to add

$$B_{21} = [eR^3F^2]_{30}, \qquad B_{22} = [eR^2D\hat{F}^2]_{24}$$

$$F_{11} = [eR^3F\bar{\psi}\psi]_{447}, \qquad F_{12} = [eR^2F\bar{\psi}_2\psi_2]_{190}$$
$$F_{13} = [eR^2DF\bar{\psi}\psi_2]_{614}, \qquad F_{14} = [eRDF\bar{\psi}_2D\psi_2]_{113}$$

The variations of this ansatz are expanded by

$$V_{11} = [eR^2DRF\bar{\epsilon}\psi]_{1563}, \qquad V_{12} = [eR^3F\bar{\epsilon}\psi_2]_{513}$$
$$V_{13} = [eR^3DF\bar{\epsilon}\psi]_{995}, \qquad V_{14} = [eRDRDF\bar{\epsilon}\psi_2]_{371}$$
$$V_{15} = [eR^2DF\bar{\epsilon}D\psi_2]_{332}, \qquad V_{16} = [eR^2DDF\bar{\epsilon}\psi_2]_{151}$$

The cancellation mechanism up to O(F^2) is sketched as

$$
\begin{array}{lcl}
\delta B_1 & \sim & V_1 \oplus V_2 \oplus V_{11} \\
\delta B_{11} & \sim & V_1 \oplus V_{11} \oplus V_{13} \\
\delta F_1 & \sim & V_1 \oplus V_2 \oplus V_3 \oplus V_{12} \oplus V_{13} \\
\delta F_2 & \sim & V_2 \oplus V_3 \oplus V_{12} \oplus V_{14} \oplus V_{15} \\
\delta B_{21} & \sim & V_{11} \oplus V_{13} \\
\delta B_{22} & \sim & V_{14} \oplus V_{16} \\
\delta F_{11} & \sim & V_{11} \oplus V_{12} \oplus V_{13} \\
\delta F_{12} & \sim & V_{12} \\
\delta F_{13} & \sim & V_{13} \oplus V_{14} \oplus V_{15} \oplus V_{16} \\
\delta F_{14} & \sim & V_{14} \oplus V_{15} \oplus V_{16}
\end{array}
$$

4169 Equations among 1544 Variables

From this cancellation we obtain

$$\mathcal{L}_{\text{boson}} = \frac{1}{24} b \Big(t_8 t_8 R^4 - \frac{1}{4!}\epsilon_{11}\epsilon_{11} R^4 - \frac{1}{6}\epsilon_{11} t_8 A R^4 \Big) + [\text{R3F2}] + [\text{R2(DF)2}]$$

The structure of R^4 terms is completely determined by local supersymmetry

[R3F2] part is governed by two parameters

$$
\begin{aligned}
&+ (3a - 24b)R_{efab}R_{ijgh}R_{ijcd}F_{abcd}F_{efgh} + (2a - 48b)R_{efab}R_{ijcg}R_{ijdh}F_{abcd}F_{efgh} \\
&+ (-8a + 96b)R_{efab}R_{icjg}R_{idjh}F_{abcd}F_{efgh} + (-8a + 72b)R_{deab}R_{ijkf}R_{ijkc}F_{oabc}F_{odef} \\
&+ (\tfrac{1}{2}a - 6b)R_{cdab}R_{ijkl}R_{ijkl}F_{opab}F_{opcd} + (-12a + 96b)R_{aief}R_{bjgh}R_{ijcd}F_{abcd}F_{efgh} \\
&+ (8a - 192b)R_{aief}R_{gjbc}R_{ijdh}F_{abcd}F_{efgh} + (-16a + 192b)R_{aief}R_{gjbc}R_{idjh}F_{abcd}F_{efgh} \\
&+ (0)R_{aide}R_{jkbf}R_{ijkc}F_{oabc}F_{odef} + (-64a + 576b)R_{aide}R_{jbkf}R_{ijkc}F_{oabc}F_{odef} \\
&+ (-32a + 288b)R_{aide}R_{jkbc}R_{ijkf}F_{oabc}F_{odef} + (-16a + 192b)R_{aicd}R_{jklb}R_{ijkl}F_{opab}F_{opcd} \\
&+ (a - 12b)R_{ijcd}R_{klab}R_{ijkl}F_{opab}F_{opcd} + (0)R_{ijac}R_{klbd}R_{ijkl}F_{opab}F_{opcd} \\
&+ (8a - 96b)R_{iajc}R_{kbld}R_{ikjl}F_{opab}F_{opcd} + (16a - 144b)R_{ijde}R_{ikaf}R_{jkbc}F_{oabc}F_{odef} \\
&+ (-16a + 144b)R_{ijde}R_{iakf}R_{jkbc}F_{oabc}F_{odef} + (0)R_{ijcd}R_{ikla}R_{jklb}F_{opab}F_{opcd} \\
&+ (-8a + 96b)R_{ijcd}R_{ikla}R_{jlkb}F_{opab}F_{opcd} + (\tfrac{64}{3}a - 192b)R_{ijad}R_{ikbe}R_{jckf}F_{oabc}F_{odef} \\
&+ (-\tfrac{64}{3}a + 192b)R_{iajd}R_{ibke}R_{jckf}F_{oabc}F_{odef} + (16a - 192b)R_{ijac}R_{ikld}R_{jklb}F_{opab}F_{opcd} \\
&+ (16a)R_{ijac}R_{ikld}R_{jlkb}F_{opab}F_{opcd} + (-48a + 384b)R_{iajc}R_{ikld}R_{jklb}F_{opab}F_{opcd} \\
&+ (16a - 192b)R_{iajc}R_{ikld}R_{jlkb}F_{opab}F_{opcd} + (\tfrac{16}{3}a - 16b)R_{iajb}R_{iklm}R_{jklm}F_{opqa}F_{opqb} \\
&+ (-\tfrac{64}{3}a + 64b)R_{ijkb}R_{ilma}R_{jklm}F_{opqa}F_{opqb} + (\tfrac{32}{3}a - 32b)R_{ijkb}R_{ilma}R_{jlkm}F_{opqa}F_{opqb} \\
&+ (\tfrac{1}{3}a)R_{ijkl}R_{ijmn}R_{klmn}F_{opqr}F_{opqr} + (-\tfrac{2}{3}a)R_{ikjl}R_{imjn}R_{kmln}F_{opqr}F_{opqr}
\end{aligned}
$$

[R2(DF)2] part is governed by two parameters

$$
\begin{aligned}
&+(-16a+96b)R_{ijkl}R_{mnop}D_mF_{inoq}D_jF_{klpq}+(-16a+96b)R_{ijkl}R_{mnop}D_iF_{jmnq}D_kF_{lopq}\\
&+(8a-48b)R_{ijkl}R_{mnop}D_iF_{jmnq}D_oF_{lkpq}+(8a-48b)R_{ijkl}R_{monp}D_mF_{jinq}D_oF_{lkpq}\\
&+(-24b)R_{ijkl}R_{imno}D_mF_{nopq}D_jF_{klpq}+(-32a)R_{ijkl}R_{imno}D_jF_{mnpq}D_kF_{lopq}\\
&+(8a)R_{ijkl}R_{imno}D_jF_{mnpq}D_oF_{lkpq}+(-16a+96b)R_{ijkl}R_{imno}D_mF_{jnpq}D_kF_{lopq}\\
&+(96b)R_{ijkl}R_{imno}D_mF_{jnpq}D_oF_{lkpq}+(16a-96b)R_{ijkl}R_{imno}D_nF_{mjpq}D_kF_{lopq}\\
&+(96b)R_{ijkl}R_{imno}D_pF_{mnjq}D_pF_{lokq}+(16a-96b)R_{ikjl}R_{imno}D_mF_{jnpq}D_kF_{lopq}\\
&+(0)R_{ijkl}R_{ijmn}D_mF_{nopq}D_kF_{lopq}+(12b)R_{ijkl}R_{ijmn}D_oF_{nmpq}D_oF_{lkpq}\\
&+(32b)R_{ikjl}R_{imjn}D_mF_{nopq}D_kF_{lopq}+(-\tfrac{8}{3}a)R_{ijkl}R_{ijmn}D_kF_{mopq}D_lF_{nopq}\\
&+(\tfrac{8}{3}a-16b)R_{ijkl}R_{ijmn}D_kF_{mopq}D_nF_{lopq}+(24b)R_{ijkl}R_{ijmn}D_oF_{mkpq}D_oF_{nlpq}\\
&+(\tfrac{16}{3}a-32b)R_{ikjl}R_{imjn}D_kF_{mopq}D_lF_{nopq}+(0)R_{ikjl}R_{imjn}D_kF_{mopq}D_nF_{lopq}\\
&+(-48b)R_{ikjl}R_{imjn}D_oF_{mkpq}D_oF_{nlpq}+(-4b)R_{ijkl}R_{ijkm}D_mF_{nopq}D_lF_{nopq}\\
&+(-16b)R_{ijkl}R_{ijkm}D_nF_{mopq}D_nF_{lopq}+(b)R_{ijkl}R_{ijkl}D_mF_{nopq}D_mF_{nopq}
\end{aligned}
$$

Appropriately choosing two parameters, this part can be consistent with the result obtained by string scattering amplitudes

Vanishing theorem :

Tree and one-loop amplitudes only contribute to the R^4 terms

Proof :

In 11 dim. there is only one superinvariant which contain R^4 terms. These become tree level or one-loop terms in type IIA by Kaluza-Klein reduction. No terms more than one-loop.

11d		10d	
	↗	$e^{-2\Phi}R^4$	Sum of KK non-zero modes (Green, Vanhove)
R^4	→	R^4	KK zero mode
	×	$e^{2(g-1)\Phi}R^4 (g > 1)$	

9. Summary

Finiteness of N=8 supergravity is reviewed

Higher derivative corrections in Type II and M-theory are considered via

- **String scattering amplitude**
- **Local Supersymmetry**

4. Perturbative and Non-perturbative Terms via Duality

- **11 dim. 1-loop amplitude on a circle**

Green, Gutperle, Vanhove
Russo, Tseytlin

Let us reexamine 1-loop amplitude for 4 gravitons in the low energy limit

$$M_4^{\text{1-loop}} \sim 2\pi I\, t_8 t_8 R^4 \qquad I = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} F(\tau, \bar{\tau}, s, t, u)$$

This expression contains the integral of loop momentum, and it is implicitly included in I

$$\int d^{10}p |w|^{\frac{\alpha'}{2}p^2} = (\alpha' \tau_2)^{-\frac{10}{2}} \qquad w = \exp(2\pi i \tau)$$

In order to lift this to 11 dimensions, the sum of KK momentum should appear.

$$I \longrightarrow I = \int \frac{d\tau}{\tau^2} \sum_{m=-\infty}^{\infty} e^{-\pi\alpha'\tau m^2/R_{11}^2}$$

$$= \frac{2}{3}\Lambda^3 + \frac{1}{\pi g_s^2}\zeta(3) \qquad R_{11} = \ell_s g_s$$

• 11 dim. 1-loop amplitude on a torus

Momentum along 9th dimension should be discretized.

$$I \longrightarrow \quad I = \int \frac{d\tau}{\tau^{3/2}} \sum_{m,n} e^{-\pi\alpha'\tau(m^2/R_9^2 + n^2/R_{11}^2)}$$

$$= R_9 \Lambda^3 + \frac{\alpha'}{2\pi V^{1/2}} Z_{3/2}(\Omega, \bar{\Omega}) \qquad V = R_9 R_{11}$$

10 dimensional type IIB is realized if $R_9 \to 0$

$$I = \frac{\alpha'}{2\pi V^{1/2}} Z_{3/2}(\Omega, \bar{\Omega})$$

$Z_{3/2}(\Omega, \bar{\Omega})$ is a non-holomorphic Eisenstein series

$$Z_{3/2}(\Omega, \bar{\Omega}) = \sum_{(m,n)\neq(0,0)} \frac{\Omega_2^{3/2}}{|m + n\Omega|^3}$$

$$= 2\zeta(3) g_s^{-3/2} + \frac{2\pi^2}{3} g_s^{1/2} + \pi \sum_{k\neq 0} e^{-2\pi|k|/g_s} \ldots$$

D-instanton